

G -covariant ϕ -coordinated quasi modules for quantum vertex algebras

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Abstract

This is a paper in a series to study quantum vertex algebras and their relations with various quantum algebras. In this paper, we introduce a notion of T -type quantum vertex algebra and a notion of G -covariant ϕ -coordinated quasi module for a T -type quantum vertex algebra with an automorphism group G . We refine and extend several previous results and we obtain a commutator formula for G -covariant ϕ -coordinated quasi modules. As an illustrating example, we study a special case of the deformed Virasoro algebra $\text{Vir}_{p,q}$ with $q = -1$, to which we associate a Clifford vertex superalgebra and its G -covariant ϕ -coordinated quasi modules.

1 Introduction

Previously, inspired by Etingof-Kazhdan's theory of quantum vertex operator algebras (see [EK]), we developed a theory of (weak) quantum vertex algebras and their modules in [Li2] and [Li3]. It was shown (see [Li3], [KL]) that quantum vertex algebras in this sense can be associated to Zamolodchikov-Faddeev algebras of a certain type. To associate more (quantum) algebras including quantum affine algebras with quantum vertex algebras, we furthermore developed in [Li6] a theory of what we called ϕ -coordinated quasi modules for a weak quantum vertex algebra. Indeed, this new theory enables us to associate quantum affine algebras with weak quantum vertex algebras *conceptually*. It then remains to realize this conceptual association *explicitly* and to study the associated quantum vertex algebras and their modules, including their ϕ -coordinated quasi modules.

In this paper, we develop the theory of ϕ -coordinated quasi modules further and we refine and extend some of the results therein, paving the way to explicitly associate quantum vertex algebras to quantum affine algebras. Specifically, we introduce a notion of T -type quantum vertex algebra and a notion of G -covariant ϕ -coordinated quasi module for a T -type quantum vertex algebra V , where G is an automorphism group of V equipped with a linear character χ . In [Li6], a Jacobi-type identity for ϕ -coordinated modules over a weak quantum vertex algebra was obtained. Among the main results of this paper, we establish a commutator formula for G -covariant ϕ -coordinated *quasi* modules for a T -type weak quantum vertex algebra. As an illustrating example, we apply this theory to the deformed Virasoro

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algebra $\mathcal{V}ir_{p,q}$ with $q = -1$ and we obtain a natural connection of this algebra with a concrete Clifford vertex superalgebra.

Now, we give a more detailed account of this paper. First, the notion of quantum vertex algebra formulated in [Li2] was somewhat derived from Etingof-Kazhdan's notion of quantum vertex operator algebra. Quantum vertex algebras in this sense are generalizations of vertex algebras and vertex super-algebras, whereas quantum vertex operator algebras in the sense of Etingof-Kazhdan are formal deformations of vertex algebras. Among the main ingredients for a quantum vertex algebra V is a rational quantum Yang-Baxter operator $\mathcal{S}(x)$ which governs a generalized commutativity (namely locality) of the associated vertex operators on V . On the one hand, this notion singles out of a large family of vertex algebra-like structures, and on the other hand, this theory is not much more complicated than the ordinary vertex algebra theory.

In regard to the theory of ϕ -coordinated quasi modules for weak quantum vertex algebras, this very parameter ϕ is what was called an associate of the one-dimensional additive formal group (law) $F(x, y) = x + y$, where an associate is a formal series $\phi(x, z) \in \mathbb{C}((x))[[z]]$, satisfying

$$\phi(x, 0) = x \quad \text{and} \quad \phi(x, \phi(y, z)) = \phi(x + y, z) \quad (= \phi(F(x, y), z)). \quad (1.1)$$

It was shown that every associate (of the additive formal group law) can be explicitly constructed by $\phi(x, z) = e^{z p(x) \frac{d}{dx}} \cdot x$ with $p(x) \in \mathbb{C}((x))$. In particular, taking $p(x) = 1$ one gets $\phi(x, z) = x + z = F(x, z)$ (the formal group itself) and taking $p(x) = x$ one gets $\phi(x, z) = x e^z$. The truth is that the ordinary theory of (weak) quantum vertex algebras and modules is governed by the formal group law, whereas to each associate ϕ one can attach a theory of ϕ -coordinated quasi modules. In the theory of weak quantum vertex algebras and modules, an important role was played by the associativity

$$Y(u, x_0 + x_2)Y(v, x_2) = Y(Y(u, x_0)v, x_2) \quad (1.2)$$

(*unrigorous*), which is commonly referred to as operator product expansion by physicists. In contrast, in the theory of ϕ -coordinated (quasi) modules, this associativity is replaced with

$$Y(u, \phi(x_2, x_0))Y(v, x_2) = Y(Y(u, x_0)v, x_2) \quad (1.3)$$

(*unrigorous*). For most of the better known algebras including quantum affine algebras, we take $\phi(x, z) = x e^z$. It was proved therein that for any highest weight module W for a quantum affine algebra of a fixed level, the Drinfeld generating functions, viewed as fields on W , generate a weak quantum vertex algebra in a certain natural way with W as a canonical ϕ -coordinated quasi module. In this way, quantum affine algebras were tied to weak quantum vertex algebras conceptually.

Recall that a rational quantum Yang-Baxter operator is an important ingredient for a quantum vertex algebra. In this paper, we define a *T-type quantum vertex algebra* to be a quantum vertex algebra V such that the rational quantum Yang-Baxter

operator $\mathcal{S}(x)$ is associated to a trigonometric quantum Yang-Baxter operator $T(x)$ with $\mathcal{S}(x) = T(e^x)$. Let V be a T-type quantum vertex algebra and let G be an automorphism group equipped with a linear character $\chi : G \rightarrow \mathbb{C}^\times$. We then define a notion of G -covariant ϕ -coordinated quasi V -module. In the definition, in addition to the associativity (1.3) with $\phi(x_2, x_0) = x_2 e^{x_0}$, two defining properties are that

$$Y_W(gv, x) = Y_W(v, \chi(g)x) \quad \text{for } g \in G, v \in V$$

and that for any $u, v \in V$, there exists a polynomial $p(x)$ with only zeroes in $\{\chi(g) \mid g \in G\}$ such that

$$p(x/z)Y_W(u, x)Y_W(v, z) \in \text{Hom}(W, W((x, z))).$$

There is a similarity between this notion and the notion of twisted V -module (see [FLM], [FFR], [D]) for a vertex algebra V , where a twisted V -module is associated to an automorphism σ of finite order N with a choice of an N -th root of unity, which amounts to choosing a linear character for $\langle \sigma \rangle = \mathbb{Z}_N$. (For a vertex operator algebra V , a notion of \mathbb{Z}_N -covariant quasi module was introduced and a canonical isomorphism between the category of \mathbb{Z}_N -twisted modules and the category of \mathbb{Z}_N -covariant quasi modules was established in [Li5].) As one of the main results of this paper, we obtain a commutator formula, which is somewhat analogous to that for twisted modules.

In this paper, as an illustrating example we apply this theory to the deformed Virasoro algebra $\mathcal{V}ir_{p,q}$ with $q = -1$. The deformed Virasoro algebra $\mathcal{V}ir_{p,q}$ with nonzero complex parameters p and q was introduced in [SKAO] and the special case with $q = -1$ was later studied in [BP]. The deformed Virasoro algebra is among the simplest quantum algebras in the sense that it is generated by one single field. We here establish a canonical isomorphism between the category of highest weight modules for $\mathcal{V}ir_{p,q}$ with $q = -1$ and the category of G -covariant ϕ -coordinated quasi modules for a specific Clifford vertex super-algebra.

As for quantum affine algebras $U_q(\hat{\mathfrak{g}})$ (with q a nonzero complex number), our speculation is that for every $\ell \in \mathbb{C}$, we shall have a canonical T-type quantum vertex algebra with an automorphism group G isomorphic to a subgroup of \mathbb{C}^\times such that highest weight modules for $U_q(\hat{\mathfrak{g}})$ of level ℓ are exactly G -covariant ϕ -coordinated quasi modules for the conjectured T -type quantum vertex algebra.

In a pioneer work [FR], E. Frenkel and N. Reshetikhin introduced a theory of deformed chiral algebras. Among the key ingredients in the notion of deformed chiral algebra are two spaces, called the space of states and the space of fields, and a *trigonometric* quantum Yang-Baxter operator on the space of fields. This theory and the Etingof-Kazhdan theory look quite different and they have been studied independently. Now, the structure of a deformed chiral algebra can be enhanced to a T -type quantum vertex algebra V equipped with an automorphism group G together with a G -covariant ϕ -coordinated quasi module.

This paper is organized as follows: Sections 2 and 3 are preliminary; In Section 2 we present some basic results on formal calculus, and in Section 3 we recall the

basic notions and results about (weak) quantum vertex algebras. In Section 4, we define the notions of T -type quantum vertex algebra and G -covariant ϕ -coordinated quasi modules, and we present several results including the commutator formula. In Section 5, we study the deformed Virasoro algebra $\mathcal{V}ir_{p,q}$ with $q = -1$.

2 Some results on formal calculus

In this section, we shall review some formal variable notations and conventions, and we formulate (and prove) certain basic results which we need in later sections.

First of all, throughout this paper all vector spaces are assumed to be over \mathbb{C} (the field of complex numbers) and \mathbb{N} denotes the set of nonnegative integers. Letters x, y, z, x_i, y_i, z_i with $i = 0, 1, \dots$ will be mutually commuting independent formal variables.

For a vector space U , $U[[x, x^{-1}]]$ denotes the space of formal infinite integer power series with coefficients in U , $U[[x]]$ denotes the space of formal infinite nonnegative power series, and $U((x))$ denotes the space of lower truncated formal Laurent series. In particular, we have a vector space $\mathbb{C}[[x, x^{-1}]]$, an algebra $\mathbb{C}[[x]]$, and a field $\mathbb{C}((x))$.

We denote by $F_{\mathbb{C}[[x]]}$ the field of fractions of $\mathbb{C}[[x]]$, which is naturally isomorphic to $\mathbb{C}((x))$. Similarly, denote by $F_{\mathbb{C}[[x_1, x_2]]}$ the field of fractions of $\mathbb{C}[[x_1, x_2]]$. We define

$$\iota_{x_1, x_2} : F_{\mathbb{C}[[x_1, x_2]]} \rightarrow \mathbb{C}((x_1))((x_2)) \quad (2.1)$$

to be the canonical extension of the embedding of $\mathbb{C}[[x_1, x_2]]$ into $\mathbb{C}((x_1))((x_2))$. Following [FLM] and [FHL], we use the following formal variable convention: For $n \in \mathbb{Z}$,

$$(x_1 - x_2)^n = \iota_{x_1, x_2}((x_1 - x_2)^n) = \sum_{i \geq 0} \binom{n}{i} (-1)^i x_1^{n-i} x_2^i. \quad (2.2)$$

For $A(x) = \sum_{n \in \mathbb{Z}} A(n)x^n \in U[[x, x^{-1}]]$ with U a vector space, we have

$$e^{z \frac{d}{dx}} A(x) = A(x+z) = \sum_{n \in \mathbb{Z}} A(n)(x+z)^n,$$

$$e^{z(x \frac{d}{dx})} A(x) = A(xe^z).$$

We also use the following formal delta functions:

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n,$$

$$x_1^{-1} \delta\left(\frac{x_2}{x_1}\right) = \sum_{n \in \mathbb{Z}} x_1^{-n-1} x_2^n,$$

$$x_0^{-1} \delta\left(\frac{x_1 - x_2}{x_0}\right) = \sum_{n \in \mathbb{Z}} x_0^{-n-1} (x_1 - x_2)^n = \sum_{n \in \mathbb{Z}} \sum_{i \geq 0} \binom{n}{i} (-1)^i x_0^{-n-1} x_1^{n-i} x_2^i.$$

For $n \geq 0$, we have

$$\frac{1}{n!} \left(\frac{\partial}{\partial x_2} \right)^n x_1^{-1} \delta \left(\frac{x_2}{x_1} \right) = (x_1 - x_2)^{-n-1} - (-x_2 + x_1)^{-n-1}. \quad (2.3)$$

An important property is the following substitution rule:

$$f(x_1, x_2) \delta \left(\frac{x_2}{x_1} \right) = f(x_2, x_2) \delta \left(\frac{x_2}{x_1} \right), \quad (2.4)$$

where $f(x_1, x_2)$ is any series such that $f(x_2, x_2)$ exists, e.g., $f(x_1, x_2) \in U((x_1, x_2))$.

We shall frequently use the following simple facts:

Lemma 2.1. *Let $\lambda \in \mathbb{C}^\times$ and let k be a positive integer. Then*

$$(x_1 - \lambda x_2)^k \left(x_2 \frac{\partial}{\partial x_2} \right)^j \delta \left(\frac{\lambda x_2}{x_1} \right) = 0 \quad (2.5)$$

for $0 \leq j \leq k - 1$.

Proof. Using another independent formal variable z , we have

$$\begin{aligned} & (x_1 - \lambda x_2)^k \sum_{j \geq 0} \frac{z^j}{j!} \left(x_2 \frac{\partial}{\partial x_2} \right)^j \delta \left(\frac{\lambda x_2}{x_1} \right) \\ &= (x_1 - \lambda x_2)^k e^{z x_2 \frac{\partial}{\partial x_2}} \delta \left(\frac{\lambda x_2}{x_1} \right) \\ &= (x_1 - \lambda x_2)^k \delta \left(\frac{\lambda x_2 e^z}{x_1} \right) \\ &= (\lambda x_2)^k (e^z - 1)^k \delta \left(\frac{\lambda x_2 e^z}{x_1} \right) \\ &\in z^k \mathbb{C}[[x_1^{\pm 1}, x_2^{\pm 1}, z]]. \end{aligned}$$

Then it follows immediately. \square

Lemma 2.2. *Let $\lambda_1, \dots, \lambda_r$ be distinct nonzero complex numbers, let k be a non-negative integer, and let $A_{ij}(x) \in U[[x, x^{-1}]]$ for $1 \leq i \leq r$, $0 \leq j \leq k$, where U is a vector space. Then*

$$\sum_{i=1}^r \sum_{j=0}^k A_{ij}(x_2) \left(x_2 \frac{\partial}{\partial x_2} \right)^j \delta \left(\frac{\lambda_i x_2}{x_1} \right) = 0 \quad (2.6)$$

if and only if $A_{ij}(x) = 0$ for all i, j .

Proof. We only need to prove the “only if” part. For $1 \leq i \leq r$, set

$$p_i(x) = \frac{(x - \lambda_1)^{k+1} \cdots (x - \lambda_r)^{k+1}}{(x - \lambda_i)^{k+1}} \in \mathbb{C}[x].$$

Then there exist $q_1(x), \dots, q_r(x) \in \mathbb{C}[x]$ such that

$$p_1(x)q_1(x) + \cdots + p_r(x)q_r(x) = 1. \quad (2.7)$$

Let $1 \leq s \leq r$ be arbitrarily fixed. For $1 \leq i \leq r$ with $i \neq s$, as $(x - \lambda_i)^{k+1}$ divides $p_s(x)$, by Lemma 2.1 we have

$$p_s(x_1/x_2) \left(x_2 \frac{\partial}{\partial x_2} \right)^j \delta \left(\frac{\lambda_i x_2}{x_1} \right) = 0 \quad \text{for } 0 \leq j \leq k.$$

Then multiplying both sides of (2.6) by $p_s(x_1/x_2)q_s(x_1/x_2)$, we get

$$\sum_{j=0}^k A_{sj}(x_2) p_s(x_1/x_2) q_s(x_1/x_2) \left(x_2 \frac{\partial}{\partial x_2} \right)^j \delta \left(\frac{\lambda_s x_2}{x_1} \right) = 0.$$

Noticing that for $1 \leq n \leq r$ with $n \neq s$ and for $0 \leq j \leq k$,

$$p_n(x_1/x_2) q_n(x_1/x_2) \left(x_2 \frac{\partial}{\partial x_2} \right)^j \delta \left(\frac{\lambda_s x_2}{x_1} \right) = 0,$$

then using (2.7) we obtain

$$\sum_{j=0}^k A_{sj}(x_2) \left(x_2 \frac{\partial}{\partial x_2} \right)^j \delta \left(\frac{\lambda_s x_2}{x_1} \right) = 0,$$

which amounts to

$$\sum_{j=0}^k A_{sj}(x_2) \left(x_2 \frac{\partial}{\partial x_2} \right)^j \delta \left(\frac{x_2}{x_1} \right) = 0. \quad (2.8)$$

For any nonzero integer m , by extracting the coefficient of x_1^{-m} we get

$$\sum_{j=0}^k m^j A_{sj}(x_2) = 0.$$

Taking $m = 1, 2, \dots, k+1$ and solving the system of equations we obtain $A_{sj}(x) = 0$ for $0 \leq j \leq k$, as desired. \square

Let W be a vector space. Set

$$\mathcal{E}(W) = \text{Hom}(W, W((x))) \subset (\text{End} W)[[x, x^{-1}]]. \quad (2.9)$$

Lemma 2.3. Let $a(x), b(x) \in \mathcal{E}(W)$, $K(x_1, x_2) \in \text{Hom}(W, W((x_2))((x_1)))$,

$$p(x) = (x - \lambda_1)^{k_1} \cdots (x - \lambda_r)^{k_r} \in \mathbb{C}[x],$$

where $\lambda_1, \dots, \lambda_r$ are distinct nonzero complex numbers and k_1, \dots, k_r are positive integers. Then

$$p(x_1/x_2) (a(x_1)b(x_2) - K(x_1, x_2)) = 0 \quad (2.10)$$

if and only if

$$a(x_1)b(x_2) - K(x_1, x_2) = \sum_{i=1}^r \sum_{j=0}^{k_i-1} A_{ij}(x_2) \left(x_2 \frac{\partial}{\partial x_2} \right)^j \delta \left(\lambda_i \frac{x_2}{x_1} \right) \quad (2.11)$$

for some $A_{ij}(x) \in \mathcal{E}(W)$, which are uniquely determined.

Proof. The uniqueness follows from Lemma 2.2, so it remains to prove the existence. From assumption we have

$$p(x_1/x_2)a(x_1)b(x_2) = p(x_1/x_2)K(x_1, x_2),$$

which implies

$$p(x_1/x_2)a(x_1)b(x_2), \quad p(x_1/x_2)K(x_1, x_2) \in \text{Hom}(W, W((x_1, x_2))).$$

Thus

$$\begin{aligned} a(x_1)b(x_2) &= \iota_{x_1, x_2} \left(\frac{1}{p(x_1/x_2)} \right) A(x_1, x_2), \\ K(x_1, x_2) &= \iota_{x_2, x_1} \left(\frac{1}{p(x_1/x_2)} \right) A(x_1, x_2) \end{aligned}$$

for some $A(x_1, x_2) \in \text{Hom}(W, W((x_1, x_2)))$. Then

$$a(x_1)b(x_2) - K(x_1, x_2) = \left(\iota_{x_1, x_2} \left(\frac{1}{p(x_1/x_2)} \right) - \iota_{x_2, x_1} \left(\frac{1}{p(x_1/x_2)} \right) \right) A(x_1, x_2).$$

Write

$$\frac{1}{p(x)} = \sum_{i=1}^r \sum_{j=0}^{k_i} \frac{a_{ij}}{(x - \lambda_i)^j}$$

with $a_{ij} \in \mathbb{C}$. We then have

$$\begin{aligned} & a(x_1)b(x_2) - K(x_1, x_2) \\ &= A(x_1, x_2) \left(\iota_{x_1, x_2} \left(\frac{1}{p(x_1/x_2)} \right) - \iota_{x_2, x_1} \left(\frac{1}{p(x_1/x_2)} \right) \right) \\ &= A(x_1, x_2) \sum_{i=1}^r \sum_{j=1}^{k_i} \left(\frac{a_{ij} x_2^j}{(x_1 - \lambda_i x_2)^j} - \frac{a_{ij} x_2^j}{(-\lambda_i x_2 + x_1)^j} \right) \\ &= \sum_{i=1}^r \sum_{j=1}^{k_i} a_{ij} \frac{1}{(j-1)!} \lambda_i^{1-j} x_2^j A(x_1, x_2) \left(\frac{\partial}{\partial x_2} \right)^{j-1} x_1^{-1} \delta \left(\frac{\lambda_i x_2}{x_1} \right). \quad (2.12) \end{aligned}$$

It follows from induction that for any nonnegative integer n and for any $B(x_1, x_2) \in \text{Hom}(W, W((x_1, x_2)))$, we have

$$B(x_1, x_2) \left(\frac{\partial}{\partial x_2} \right)^n x_1^{-1} \delta \left(\frac{\lambda_i x_2}{x_1} \right) = \sum_{i=0}^n B_i(x_2) \left(\frac{\partial}{\partial x_2} \right)^i \delta \left(\frac{\lambda_i x_2}{x_1} \right) \quad (2.13)$$

for some $B_i(x) \in \mathcal{E}(W)$. On the other hand, for $n \geq 1$ we have

$$\left(\frac{\partial}{\partial x_2} \right)^n = x_2^{-n} \left(x_2 \frac{\partial}{\partial x_2} \right) \left(x_2 \frac{\partial}{\partial x_2} - 1 \right) \cdots \left(x_2 \frac{\partial}{\partial x_2} - (n-1) \right). \quad (2.14)$$

Using all these we get

$$a(x_1)b(x_2) - K(x_1, x_2) = \sum_{i=1}^r \sum_{j=0}^{k_i-1} A_{ij}(x_2) \left(x_2 \frac{\partial}{\partial x_2} \right)^j \delta \left(\frac{\lambda_i x_2}{x_1} \right)$$

for some $A_{ij}(x) \in \mathcal{E}(W)$, as desired. \square

We shall also need the following simple fact:

Lemma 2.4. *Let W be a vector space, let $A(x_1, x_2) \in \text{Hom}(W, W((x_1, x_2)))$, and let $\lambda \in \mathbb{C}^\times$. If $A(x_1, x_2) \neq 0$, then*

$$A(x_1, x_2) = (x_1 - \lambda x_2)^k B(x_1, x_2)$$

for some $k \in \mathbb{N}$, $B(x_1, x_2) \in \text{Hom}(W, W((x_1, x_2)))$ with $B(\lambda x_2, x_2) \neq 0$.

Proof. Notice that for any $m \in \mathbb{Z}$,

$$x_1^m - (\lambda x_2)^m = (x_1 - \lambda x_2) F_m(x_1, x_2)$$

for some $F_m(x_1, x_2) \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$. For any $C(x_1, x_2) \in \text{Hom}(W, W((x_1, x_2)))$, if $C(\lambda x_2, x_2) = 0$, then

$$C(x_1, x_2) = C(x_1, x_2) - C(\lambda x_2, x_2) = (x_1 - \lambda x_2) \bar{C}(x_1, x_2)$$

for some $\bar{C}(x_1, x_2) \in \text{Hom}(W, W((x_1, x_2)))$. On the other hand, as $A(x_1, x_2) \neq 0$, $A(x_1, x_2)w \neq 0$ for some $w \in W$, where $A(x_1, x_2)w \in W((x_1, x_2))$. Then $A(x_1, x_2)w = x_1^r x_2^s G(x_1, x_2)$ for some $r, s \in \mathbb{Z}$, $G(x_1, x_2) \in W[[x_1, x_2]]$ (nonzero). Assuming $G(x_1 + \lambda x_2, x_2) = \sum_{m \geq 0} G_m(x_2) x_1^m$ with $G_m(x_2) \in \mathbb{C}[[x_2]]$, we have

$$G(x_1, x_2) = \sum_{m \geq 0} G_m(x_2) (x_1 - \lambda x_2)^m.$$

It follows that $G(x_1, x_2) = (x_1 - \lambda x_2)^n H(x_1, x_2)$ with $n \in \mathbb{N}$ and $H(x_1, x_2) \in W[[x_1, x_2]]$ such that $H(\lambda x_2, x_2) \neq 0$. Then

$$A(x_1, x_2)w = (x_1 - \lambda x_2)^n x_1^r x_2^s H(x_1, x_2).$$

In view of these, there exists a largest nonnegative integer k such that $A(x_1, x_2) = (x_1 - \lambda x_2)^k B(x_1, x_2)$ for some $B(x_1, x_2) \in \text{Hom}(W, W((x_1, x_2)))$ with $B(\lambda x_2, x_2) \neq 0$. \square

3 Quantum vertex algebras and their modules

In this section, we recall the notions of (weak) quantum vertex algebras and their modules, including a conceptual construction.

We first recall the notion of weak quantum vertex algebra, which was formulated and studied in [Li2] and [Li3].

Definition 3.1. A *weak quantum vertex algebra* is a vector space V equipped with a linear map

$$\begin{aligned} Y(\cdot, x) : V &\rightarrow \text{Hom}(V, V((x))) \subset (\text{End}V)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \quad (\text{where } v_n \in \text{End}V), \end{aligned}$$

called the *adjoint vertex operator map*, and a vector $\mathbf{1} \in V$, called the *vacuum vector*, satisfying the following conditions: For $v \in V$,

$$Y(\mathbf{1}, x)v = v, \quad Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \rightarrow 0} Y(v, x)\mathbf{1} = v,$$

and for $u, v \in V$, there exist

$$u^{(i)}, v^{(i)} \in V, \quad f_i(x) \in \mathbb{C}((x)) \quad \text{for } i = 1, \dots, r$$

such that

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y(u, x_1) Y(v, x_2) \\ & - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \sum_{i=1}^r f_i(x_2 - x_1) Y(v^{(i)}, x_2) Y(u^{(i)}, x_1) \\ & = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2) \end{aligned} \tag{3.1}$$

(the \mathcal{S} -Jacobi identity).

The following was proved in [Li2]:

Proposition 3.2. *In Definition 3.1, the \mathcal{S} -Jacobi identity axiom can be equivalently replaced by weak associativity: For $u, v, w \in V$, there exists a nonnegative integer l such that*

$$(x_0 + x_2)^l Y(u, x_0 + x_2) Y(v, x_2) w = (x_0 + x_2)^l Y(Y(u, x_0)v, x_2) w, \tag{3.2}$$

and \mathcal{S} -locality: For any $u, v \in V$, there exist

$$u^{(i)}, v^{(i)} \in V, \quad f_i(x) \in \mathbb{C}((x)) \quad \text{for } i = 1, \dots, r,$$

and a nonnegative integer k such that

$$(x_1 - x_2)^k Y(u, x_1) Y(v, x_2) = (x_1 - x_2)^k \sum_{i=1}^r f_i(x_2 - x_1) Y(v^{(i)}, x_2) Y(u^{(i)}, x_1). \tag{3.3}$$

Let V be a weak quantum vertex algebra. Define a linear operator \mathcal{D} on V by $\mathcal{D}(v) = v_{-2}\mathbf{1}$ for $v \in V$. Then

$$[\mathcal{D}, Y(v, x)] = Y(\mathcal{D}(v), x) = \frac{d}{dx}Y(v, x) \quad (3.4)$$

for $v \in V$. It was proved in [Li2] that the \mathcal{S} -locality relation (3.3) amounts to

$$Y(u, x)v = \sum_{i=1}^r f_i(-x)e^{x\mathcal{D}}Y(v^{(i)}, -x)u^{(i)} \quad (3.5)$$

(the \mathcal{S} -skew symmetry).

Definition 3.3. Let V be a weak quantum vertex algebra. A V -module is a vector space W equipped with a linear map

$$Y_W(\cdot, x) : V \rightarrow \text{Hom}(W, W((x))) \subset (\text{End}W)[[x, x^{-1}]],$$

satisfying the conditions that $Y_W(\mathbf{1}, x) = 1_W$ (the identity operator on W) and that for any $u, v \in V$, $w \in W$, there exists a nonnegative integer l such that

$$(x_0 + x_2)^l Y_W(u, x_0 + x_2)Y_W(v, x_2)w = (x_0 + x_2)^l Y_W(Y(u, x_0)v, x_2)w. \quad (3.6)$$

We have (see [Li2]):

Proposition 3.4. Let V be a weak quantum vertex algebra and let (W, Y_W) be a V -module. Then

$$Y_W(\mathcal{D}v, x) = \frac{d}{dx}Y_W(v, x) \quad \text{for } v \in V. \quad (3.7)$$

For $u, v \in V$, let $u^{(i)}, v^{(i)} \in V$, $f_i(x) \in \mathbb{C}((x))$ ($i = 1, \dots, r$) such that \mathcal{S} -Jacobi identity (3.1) holds. Then

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y_W(u, x_1)Y_W(v, x_2) \\ & - x_0^{-1}\delta\left(\frac{x_2 - x_1}{-x_0}\right)\sum_{i=1}^r f_i(x_2 - x_1)Y_W(v^{(i)}, x_2)Y_W(u^{(i)}, x_1) \\ & = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y_W(Y(u, x_0)v, x_2). \end{aligned} \quad (3.8)$$

We next recall from [Li2] the conceptual construction of weak quantum vertex algebras and modules.

Definition 3.5. Let W be a vector space and let $a(x), b(x) \in \mathcal{E}(W)$. The ordered pair $(a(x), b(x))$ is said to be *compatible* if there exists $k \in \mathbb{N}$ such that

$$(x_1 - x_2)^k a(x_1)b(x_2) \in \text{Hom}(W, W((x_1, x_2))). \quad (3.9)$$

Let $(a(x), b(x))$ be a compatible pair in $\mathcal{E}(W)$. Define $a(x)_n b(x) \in \mathcal{E}(W)$ for $n \in \mathbb{Z}$ in terms of generating function

$$Y_{\mathcal{E}}(a(x), z)b(x) = \sum_{n \in \mathbb{Z}} a(x)_n b(x) z^{-n-1}$$

by

$$Y_{\mathcal{E}}(a(x), z)b(x) = z^{-k} \left((x_1 - x)^k a(x_1)b(x) \right) |_{x_1=x+z}, \quad (3.10)$$

where k is any nonnegative integer such that (3.9) holds.

Definition 3.6. A subset U of $\mathcal{E}(W)$ is said to be \mathcal{S} -local if for any $a(x), b(x) \in U$, there exist

$$a^{(i)}(x), b^{(i)}(x) \in U, \quad f_i(x) \in \mathbb{C}((x)) \quad (i = 1, \dots, r)$$

such that

$$(x_1 - x_2)^k a(x_1)b(x_2) = (x_1 - x_2)^k \sum_{i=1}^r f_i(x_2 - x_1) b^{(i)}(x_2) a^{(i)}(x_1) \quad (3.11)$$

for some nonnegative integer k .

Note that if U is an \mathcal{S} -local subset of $\mathcal{E}(W)$, then every pair $(a(x), b(x))$ in U is compatible. The following result was obtained in [Li2]:

Theorem 3.7. *Every \mathcal{S} -local subset U of $\mathcal{E}(W)$ generates a weak quantum vertex algebra $\langle U \rangle$ with W as a canonical module with $Y_W(\alpha(x), z) = \alpha(z)$ for $\alpha(x) \in \langle U \rangle$.*

Let U be a vector space. Recall that a *rational quantum Yang-Baxter operator* on U is a linear map

$$\mathcal{S}(x) : U \otimes U \rightarrow U \otimes U \otimes \mathbb{C}((x)),$$

satisfying

$$\mathcal{S}_{12}(x)\mathcal{S}_{13}(x+z)\mathcal{S}_{23}(z) = \mathcal{S}_{23}(z)\mathcal{S}_{13}(x+z)\mathcal{S}_{12}(x) \quad (3.12)$$

(the *quantum Yang-Baxter equation*), where for $1 \leq i < j \leq 3$,

$$\mathcal{S}^{ij}(x) : U \otimes U \otimes U \rightarrow U \otimes U \otimes U \otimes \mathbb{C}((x))$$

denotes the canonical extension of $\mathcal{S}(x)$. It is said to be *unitary* if

$$\mathcal{S}(x)\mathcal{S}^{21}(-x) = 1,$$

where $\mathcal{S}^{21}(x) = \sigma \mathcal{S}(x) \sigma$ with σ denoting the flip operator on $U \otimes U$.

For a weak quantum vertex algebra V , following [EK] let

$$Y(x) : V \otimes V \rightarrow V((x)) \quad (3.13)$$

be the canonical linear map associated to the vertex operator map $Y(\cdot, x)$.

The following notion, which was formulated in [Li2], was derived from Etingof-Kazhdan's notion of quantum vertex operator algebra (see [EK]):

Definition 3.8. A *quantum vertex algebra* is a weak quantum vertex algebra V equipped with a unitary rational quantum Yang-Baxter operator $\mathcal{S}(x)$ on V , satisfying the following conditions:

$$\mathcal{S}(x)(\mathbf{1} \otimes v) = \mathbf{1} \otimes v \quad \text{for } v \in V, \quad (3.14)$$

$$[\mathcal{D} \otimes \mathbf{1}, \mathcal{S}(x)] = -\frac{d}{dx}\mathcal{S}(x), \quad (3.15)$$

$$Y(u, x)v = e^{x\mathcal{D}}Y(-x)\mathcal{S}(-x)(v \otimes u) \quad \text{for } u, v \in V, \quad (3.16)$$

$$\mathcal{S}(x_1)(Y(x_2) \otimes \mathbf{1}) = (Y(x_2) \otimes \mathbf{1})\mathcal{S}^{23}(x_1)\mathcal{S}^{13}(x_1 + x_2), \quad (3.17)$$

where \mathcal{D} is the linear operator on V defined by $\mathcal{D}(v) = v_{-2}\mathbf{1}$ for $v \in V$. We sometimes denote a quantum vertex algebra by a pair (V, \mathcal{S}) .

The following notion was due to Etingof-Kazhdan (see [EK]):

Definition 3.9. Let V be a weak quantum vertex algebra. For any positive integer n , define a linear map

$$Z_n : V^{\otimes n} \otimes \mathbb{C}((x_1))((x_2)) \cdots ((x_n)) \rightarrow V((x_1))((x_2)) \cdots ((x_n))$$

by

$$Z_n(v^{(1)} \otimes \cdots \otimes v^{(n)} \otimes f) = fY(v^{(1)}, x_1) \cdots Y(v^{(n)}, x_n)\mathbf{1}$$

for $v^{(1)}, \dots, v^{(n)} \in V$, $f \in \mathbb{C}((x_1))((x_2)) \cdots ((x_n))$. V is said to be *non-degenerate* if for every positive integer n , Z_n is injective.

The following result can be found in [Li2] (cf. [EK]):

Proposition 3.10. *Let V be a weak quantum vertex algebra. Assume that V is non-degenerate. Then there exists a linear map $\mathcal{S}(x) : V \otimes V \rightarrow V \otimes V \otimes \mathbb{C}((x))$, which is uniquely determined by*

$$Y(u, x)v = e^{x\mathcal{D}}Y(-x)\mathcal{S}(-x)(v \otimes u) \quad \text{for } u, v \in V.$$

Furthermore, (V, \mathcal{S}) carries the structure of a quantum vertex algebra and the following relation holds

$$[1 \otimes \mathcal{D}, \mathcal{S}(x)] = \frac{d}{dx}\mathcal{S}(x). \quad (3.18)$$

Remark 3.11. Note that a quantum vertex algebra was defined as a pair (V, \mathcal{S}) . In view of Proposition 3.10, the term “a non-degenerate quantum vertex algebra” without reference to a quantum Yang-Baxter operator is unambiguous. If a weak quantum vertex algebra V is of countable dimension over \mathbb{C} and if V as a V -module is irreducible, then V is non-degenerate by a result of [Li3] (Corollary 3.10). In view of this, the term “irreducible quantum vertex algebra” is also unambiguous.

We conclude this section with some basic notions. The notions of homomorphism, isomorphism and automorphism for weak quantum vertex algebras are defined in the obvious way. For example, an *automorphism* of a weak quantum vertex algebra V is a bijective linear endomorphism σ of V such that $\sigma(\mathbf{1}) = \mathbf{1}$ and

$$\sigma(Y(u, x)v) = Y(\sigma(u), x)\sigma(v) \quad \text{for } u, v \in V.$$

Definition 3.12. A subset U of a weak quantum vertex algebra V is called a *generating subset* if V is linearly spanned by vectors

$$u_{n_1}^{(1)} \cdots u_{n_r}^{(r)} \mathbf{1}$$

for $r \in \mathbb{N}$, $u^{(i)} \in U$, $n_i \in \mathbb{Z}$ ($1 \leq i \leq r$).

4 ϕ -coordinated quasi modules for weak quantum vertex algebras

In this section, we first recall from [Li6] the definition of a ϕ -coordinated quasi module for a weak quantum vertex algebra and the conceptual construction. We then define the notions of T -type weak quantum vertex algebra and G -covariant ϕ -coordinated quasi module. As one of the main results we establish a commutator formula.

We begin with a convention. Note that for any $p(x) \in \mathbb{C}[x]$, $p(e^x) \in \mathbb{C}[[x]]$. One can show (cf. [Li6]) that $p(e^x) = 0$ if and only if $p(x) = 0$. Recall that $\mathbb{C}(x)$ denotes the field of rational functions. For any $g(x) = p(x)/q(x) \in \mathbb{C}(x)$ with $p(x), q(x) \in \mathbb{C}[x]$, we define

$$g(e^x) = \frac{p(e^x)}{q(e^x)} \in F_{\mathbb{C}[[x]]} = \mathbb{C}((x)). \quad (4.1)$$

Definition 4.1. Let V be a weak quantum vertex algebra. A ϕ -coordinated quasi V -module is a vector space W equipped with a linear map

$$Y_W(\cdot, x) : V \rightarrow \text{Hom}(W, W((x))) \subset (\text{End}W)[[x, x^{-1}]],$$

satisfying the conditions that $Y_W(\mathbf{1}, x) = 1_W$ and that for $u, v \in V$, there exists a nonzero polynomial $p(x)$ such that

$$p(x_1/x_2)Y_W(u, x_1)Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))) \quad (4.2)$$

and

$$p(e^z)Y_W(Y(u, z)v, x_2) = (p(x_1/x_2)Y_W(u, x_1)Y_W(v, x_2))|_{x_1=x_2e^z}. \quad (4.3)$$

Remark 4.2. The parameter ϕ in Definition 4.1 refers to the formal series $\phi(x, z) = xe^z$, which is a particular associate of the one-dimensional additive formal group (law) $F(x, y) = x + y$, as defined in [Li6].

The following was proved in [Li6] (Proposition 5.6):

Proposition 4.3. *Let V be a weak quantum vertex algebra, let $u, v \in V$, and let (W, Y_W) be a ϕ -coordinated quasi V -module. Suppose that*

$$\begin{aligned} & (x_1 - x_2)^k Y(u, x_1) Y(v, x_2) \\ &= (x_1 - x_2)^k \sum_{i=1}^r \iota_{x_2, x_1}(f_i(e^{x_1 - x_2})) Y(v^{(i)}, x_2) Y(u^{(i)}, x_1), \end{aligned} \quad (4.4)$$

where $k \in \mathbb{N}$, $f_i(x) \in \mathbb{C}(x)$, $u^{(i)}, v^{(i)} \in V$, and suppose that $p(x)$ is any nonzero polynomial such that

$$p(x_1/x_2) Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))).$$

Then

$$\begin{aligned} & p(x_1/x_2) Y_W(u, x_1) Y_W(v, x_2) \\ &= p(x_1/x_2) \sum_{i=1}^r \iota_{x_2, x_1}(f_i(x_1/x_2)) Y_W(v^{(i)}, x_2) Y_W(u^{(i)}, x_1). \end{aligned} \quad (4.5)$$

Set

$$\log(1 + z) = \sum_{n \geq 1} (-1)^{n-1} \frac{z^n}{n} \in z\mathbb{C}[[z]]. \quad (4.6)$$

Notice that $\sum_{n \geq 1} (-1)^{n-1} \frac{z^{n-1}}{n}$ is a unit in $\mathbb{C}[[z]]$. For $m \in \mathbb{Z}$, it is understood that

$$(\log(1 + z))^m = z^m \left(\sum_{n \geq 1} (-1)^{n-1} \frac{z^{n-1}}{n} \right)^m \in z^m \mathbb{C}[[z]] \subset \mathbb{C}((z)).$$

The following was proved in [Li6] (Lemma 5.8)²:

Lemma 4.4. *Let W be a vector space and let*

$$A(x_1, x_2) \in \text{Hom}(W, W((x_1))((x_2))), \quad B(x_1, x_2) \in \text{Hom}(W, W((x_2))((x_1))),$$

$$C(x_0, x_2) \in \text{Hom}(W, W((x_2))((x_0))).$$

If there exists a nonnegative integer k such that

$$\begin{aligned} & (x_1 - x_2)^k A(x_1, x_2) = (x_1 - x_2)^k B(x_1, x_2), \\ & ((x_1 - x_2)^k A(x_1, x_2))|_{x_1=x_2 e^{x_0}} = x_2^k (e^{x_0} - 1)^k C(x_0, x_2), \end{aligned}$$

²There is a typo in [Li6]: $B(x_2, x_1)$ in (5.18) and in the proof should be $B(x_1, x_2)$.

then

$$\begin{aligned} & (zx_2)^{-1} \delta \left(\frac{x_1 - x_2}{zx_2} \right) A(x_1, x_2) - (zx_2)^{-1} \delta \left(\frac{x_2 - x_1}{-zx_2} \right) B(x_1, x_2) \\ &= x_1^{-1} \delta \left(\frac{x_2(1+z)}{x_1} \right) C(\log(1+z), x_2). \end{aligned} \quad (4.7)$$

Furthermore, the converse is also true.

The following result generalizes Proposition 5.9 of [Li6]:

Proposition 4.5. *Let V be a weak quantum vertex algebra and let (W, Y_W) be a ϕ -coordinated quasi V -module. Let $u, v \in V$. Assume*

$$u^{(i)}, v^{(i)} \in V, \quad f_i(x) \in \mathbb{C}(x) \quad (i = 1, \dots, r)$$

such that (4.4) holds for some nonnegative integer k . Then there exists a nonzero polynomial $p(x)$ such that

$$\begin{aligned} & (xz)^{-1} \delta \left(\frac{x_1 - x}{xz} \right) p(x_1/x) Y_W(u, x_1) Y_W(v, x) \\ & - (xz)^{-1} \delta \left(\frac{x - x_1}{-xz} \right) p(x_1/x) \sum_{i=1}^r \iota_{x_1, x} (f_i(x_1/x)) Y_W(v^{(i)}, x) Y_W(u^{(i)}, x_1) \\ &= x_1^{-1} \delta \left(\frac{x(1+z)}{x_1} \right) p(x_1/x) Y_W(Y(u, \log(1+z))v, x). \end{aligned} \quad (4.8)$$

Proof. From definition, there exists a nonzero polynomial $p(x)$ such that

$$p(x_1/x_2) Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))) \quad (4.9)$$

and

$$p(e^z) Y_W(Y(u, z)v, x_2) = (p(x_1/x_2) Y_W(u, x_1) Y_W(v, x_2))|_{x_1=x_2 e^z}. \quad (4.10)$$

With (4.9), by Proposition 4.3 we have

$$\begin{aligned} & p(x_1/x_2) Y_W(u, x_1) Y_W(v, x_2) \\ &= p(x_1/x_2) \sum_{i=1}^r \iota_{x_2, x_1} (f_i(x_1/x_2)) Y_W(v^{(i)}, x_2) Y_W(u^{(i)}, x_1). \end{aligned} \quad (4.11)$$

Then it follows immediately from Lemma 4.4 (with $k = 0$). \square

Motivated by Proposition 4.3 we formulate the following notion:

Definition 4.6. A weak quantum vertex algebra V is said to be of T -type if for any $u, v \in V$, there exist

$$u^{(i)}, v^{(i)} \in V, \quad f_i(x) \in \mathbb{C}(x) \quad (i = 1, \dots, r)$$

such that (4.4) holds for some nonnegative integer k .

Recall that a *trigonometric quantum Yang-Baxter operator* on a vector space U is a linear map

$$\mathcal{S}(x) : U \otimes U \rightarrow U \otimes U \otimes \mathbb{C}(x),$$

satisfying quantum Yang-Baxter equation

$$\mathcal{S}_{12}(x)\mathcal{S}_{13}(xz)\mathcal{S}_{23}(z) = \mathcal{S}_{23}(z)\mathcal{S}_{13}(xz)\mathcal{S}_{12}(x). \quad (4.12)$$

It is said to be *unitary* if $\mathcal{S}_{21}(1/x)\mathcal{S}(x) = 1$.

Definition 4.7. A quantum vertex algebra $(V, \mathcal{S}(x))$ is said to be of T -type if there exists a unitary trigonometric quantum Yang-Baxter operator $T(x)$ on V such that $\mathcal{S}(x) = T(e^x)$.

The following technical result follows from the proof of Proposition 2.6 in [LTW] with $\mathbb{C}((x))$ replaced by $\mathbb{C}(x)$ in a few places:

Lemma 4.8. *Let V be a weak quantum vertex algebra. Suppose that U is a generating subset of V such that the condition in Definition 4.6 with U in place of V holds. Then V is of T -type.*

Next, we recall from [Li6] the conceptual construction of weak quantum vertex algebras and their ϕ -coordinated quasi modules. Let W be a vector space which is fixed for a while. Let $a(x), b(x) \in \mathcal{E}(W)$. Assume that there exists a nonzero polynomial $p(x)$ such that

$$p(x/z)a(x)b(z) \in \text{Hom}(W, W((x, z))). \quad (4.13)$$

Define $a(x)_n^e b(x) \in \mathcal{E}(W)$ for $n \in \mathbb{Z}$ in terms of generating function

$$Y_{\mathcal{E}}^e(a(x), z)b(x) = \sum_{n \in \mathbb{Z}} (a(x)_n^e b(x)) z^{-n-1}$$

by

$$Y_{\mathcal{E}}^e(a(x), z)b(x) = p(e^z)^{-1} (p(x_1/x)a(x_1)b(x))|_{x_1=xe^z}, \quad (4.14)$$

where $p(x)$ is any nonzero polynomial such that (4.13) holds and $p(e^z)^{-1}$ denotes the inverse of $p(e^z)$ in $\mathbb{C}((z))$.

Definition 4.9. A subset U of $\mathcal{E}(W)$ is said to be *quasi \mathcal{S}_{trig} -local* if for any $a(x), b(x) \in U$, there exist

$$a^{(i)}(x), b^{(i)}(x) \in U, f_i(x) \in \mathbb{C}(x) \quad (i = 1, \dots, r)$$

such that

$$p(x/z)a(x)b(z) = p(x/z) \sum_{i=1}^r \iota_{z,x}(f(z/x))b^{(i)}(z)a^{(i)}(x) \quad (4.15)$$

for some nonzero polynomial $p(x)$. A subset U of $\mathcal{E}(W)$ is *quasi \mathcal{S}_{trig} -compatible* if for any $a(x), b(x) \in U$, there exists a nonzero polynomial $p(x)$ such that

$$p(x/z)a(x)b(z) \in \text{Hom}(W, W((x, z))).$$

Note that relation (4.15) implies (4.13), so that every quasi \mathcal{S}_{trig} -local subset is automatically quasi \mathcal{S}_{trig} -compatible.

The following strengthens one of the main results in [Li6]:

Theorem 4.10. *Let W be a vector space and let U be any quasi \mathcal{S}_{trig} -local subset of $\mathcal{E}(W)$. Then U generates a weak quantum vertex algebra $\langle U \rangle_e$ with W as a faithful ϕ -coordinated quasi module where $Y_W(\alpha(x), z) = \alpha(z)$ for $\alpha(x) \in \langle U \rangle_e$. Furthermore, $\langle U \rangle_e$ is of T -type.*

Proof. The first part was proved in [Li6] (Theorem 5.4). Then we have a weak quantum vertex algebra $\langle U \rangle_e$ with U as a generating subset. Let $a(x), b(x) \in U$. There exist

$$a^{(i)}(x), b^{(i)}(x) \in U, f_i(x) \in \mathbb{C}(x) \quad (i = 1, \dots, r)$$

such that

$$p(x/z)a(x)b(z) = p(x/z) \sum_{i=1}^r \iota_{z,x}(f(z/x))b^{(i)}(z)a^{(i)}(x)$$

for some nonzero polynomial $p(x)$. By Proposition 5.3 of [Li6], we have

$$\begin{aligned} & (x_1 - x_2)^k Y_{\mathcal{E}}^e(a(x), x_1) Y_{\mathcal{E}}^e(b(x), x_2) \\ &= (x_1 - x_2)^k \sum_{i=1}^r \iota_{x_2, x_1}(f_i(e^{x_2 - x_1})) Y_{\mathcal{E}}^e(b^{(i)}(x), x_2) Y_{\mathcal{E}}^e(a^{(i)}(x), x_1) \end{aligned}$$

for some nonnegative integer k . Then it follows from Lemma 4.8 that $\langle U \rangle_e$ is of T -type. \square

We have the following result (cf. [Li6], Proposition 4.11):

Proposition 4.11. *Let V be a T -type weak quantum vertex algebra and let (W, Y_W) be a ϕ -coordinated quasi V -module. Set $V_W = \{Y_W(v, x) \mid v \in V\}$. Then V_W is a quasi \mathcal{S}_{trig} -local subspace of $\mathcal{E}(W)$, $(V_W, Y_{\mathcal{E}}^e, 1_W)$ is a weak quantum vertex algebra, and Y_W is a homomorphism of weak quantum vertex algebras.*

Proof. Combining Definition 4.1 with Proposition 4.3, we see that V_W is a quasi \mathcal{S}_{trig} -local subspace of $\mathcal{E}(W)$. Let $u, v \in V$. By definition, there exists a nonzero polynomial $p(x)$ such that

$$p(x_1/x)Y_W(u, x_1)Y_W(v, x) \in \text{Hom}(W, W((x_1, x)))$$

and

$$p(e^z)Y_W(Y(u, z)v, x) = (p(x_1/x)Y_W(u, x_1)Y_W(v, x))|_{x_1=xe^z}.$$

On the other hand, from the definition of $Y_{\mathcal{E}}^e(\cdot, x)$ we have

$$p(e^z)Y_{\mathcal{E}}^e(Y_W(u, x), z)Y_W(v, x) = (p(x_1/x)Y_W(u, x_1)Y_W(v, x))|_{x_1=xe^z}.$$

Consequently,

$$p(e^z)Y_{\mathcal{E}}^e(Y_W(u, x), z)Y_W(v, x) = p(e^z)Y_W(Y(u, z)v, x).$$

Since both $Y_{\mathcal{E}}^e(Y_W(u, x), z)Y_W(v, x)$ and $Y_W(Y(u, z)v, x)$ involve only finitely many negative powers of z , by cancellation we obtain

$$Y_{\mathcal{E}}^e(Y_W(u, x), z)Y_W(v, x) = Y_W(Y(u, z)v, x).$$

It follows that $(V_W, Y_{\mathcal{E}}^e, 1_W)$ is a weak quantum vertex algebra and Y_W is a homomorphism of weak quantum vertex algebras. \square

Note that $a(\lambda x) \in \mathcal{E}(W)$ for any $\lambda \in \mathbb{C}^\times$, $a(x) \in \mathcal{E}(W)$. For $\lambda \in \mathbb{C}^\times$, define $R_\lambda \in \text{End}(\mathcal{E}(W))$ by

$$R_\lambda(a(x)) = a(\lambda x) \quad \text{for } a(x) \in \mathcal{E}(W). \quad (4.16)$$

This gives rise to a group action of \mathbb{C}^\times on $\mathcal{E}(W)$. From now on, we fix this particular action.

Proposition 4.12. *Let W be a vector space, let Γ be a subgroup of \mathbb{C}^\times , and let U be a quasi \mathcal{S}_{trig} -local subset of $\mathcal{E}(W)$ such that U is Γ -stable. Then Γ is an automorphism group of the weak quantum vertex algebra $\langle U \rangle_e$ generated by U .*

Proof. Let $a(x), b(x) \in \langle U \rangle_e$. Then there exists a nonzero polynomial $p(x)$ such that

$$p(x_1/x)a(x_1)b(x) \in \text{Hom}(W, W((x_1, x))).$$

For any nonzero complex number λ we have

$$p(x_1/x)a(\lambda x_1)b(\lambda x) \in \text{Hom}(W, W((x_1, x))).$$

Then we get

$$\begin{aligned}
& Y_{\mathcal{E}}^e(a(\lambda x), z)b(\lambda x) \\
&= p(e^z)^{-1} (p(x_1/x)a(\lambda x_1)b(\lambda x))|_{x_1=xe^z} \\
&= [p(e^z)^{-1} (p(\bar{x}_1/\bar{x})a(\bar{x}_1)b(\bar{x}))|_{\bar{x}_1=\bar{x}e^z}]|_{\bar{x}=\lambda x} \\
&= [Y_{\mathcal{E}}^e(a(\bar{x}), z)b(\bar{x})]_{\bar{x}=\lambda x}.
\end{aligned}$$

It follows from induction that $\langle U \rangle_e$ is Γ -stable and Γ acts on weak quantum vertex algebra $\langle U \rangle_e$ by automorphisms. \square

Lemma 4.13. *Let W be a vector space, let $a(x), b(x) \in \mathcal{E}(W)$, and let*

$$K(x_1, x_2) \in \text{Hom}(W, W((x_2))((x_1))).$$

Assume

$$\begin{aligned}
a(x_1)b(x_2) - K(x_1, x_2) &= \sum_{j=0}^r A_j(x_2) \frac{1}{j!} \left(x_2 \frac{\partial}{\partial x_2} \right)^j \delta \left(\frac{x_2}{x_1} \right) \\
&\quad + \sum_{i=1}^k \sum_{j=0}^s B_{ij}(x_2) \frac{1}{j!} \left(x_2 \frac{\partial}{\partial x_2} \right)^j \delta \left(\frac{\lambda_i x_2}{x_1} \right), \quad (4.17)
\end{aligned}$$

where $A_j(x), B_{ij}(x) \in \mathcal{E}(W)$, $\lambda_i \in \mathbb{C}^\times$ with $\lambda_i \neq 1$ for $1 \leq i \leq k$. Then

$$\begin{aligned}
a(x)_j^e b(x) &= A_j(x) \quad \text{for } 0 \leq j \leq r, \\
a(x)_j^e b(x) &= 0 \quad \text{for } j > r.
\end{aligned} \quad (4.18)$$

Proof. Set

$$q(x) = \prod_{i=1}^k ((x-1)^{r+1} - (\lambda_i - 1)^{r+1})^{s+1} \in \mathbb{C}[x].$$

Noticing that

$$q(x) = \bar{q}(x) \prod_{i=1}^k (x - \lambda_i)^{s+1}$$

for some $\bar{q}(x) \in \mathbb{C}[x]$, we have

$$q(x_1/x_2) \left(x_2 \frac{\partial}{\partial x_2} \right)^j \delta \left(\frac{\lambda_i x_2}{x_1} \right) = 0$$

for $0 \leq j \leq s$, $1 \leq i \leq k$. Then we get

$$q(x_1/x_2) (a(x_1)b(x_2) - K(x_1, x_2)) = \sum_{j=0}^r q(x_1/x_2) A_j(x_2) \frac{1}{j!} \left(x_2 \frac{\partial}{\partial x_2} \right)^j \delta \left(\frac{x_2}{x_1} \right) \quad (4.19)$$

This implies

$$(x_1/x_2 - 1)^{r+1} q(x_1/x_2) (a(x_1)b(x_2) - K(x_1, x_2)) = 0.$$

From definition we have

$$(e^{x_0} - 1)^{r+1} q(e^{x_0}) Y_{\mathcal{E}}^e(a(x), x_0) b(x) = ((x_1/x - 1)^{r+1} q(x_1/x) a(x_1) b(x))|_{x_1 = x e^{x_0}}.$$

With the above two identities, by Lemma 4.4 we obtain

$$\begin{aligned} & x_1^{-1} \delta \left(\frac{x(1+z)}{x_1} \right) q(x_1/x) Y_{\mathcal{E}}^e(a(x), \log(1+z)) b(x) \\ &= (xz)^{-1} \delta \left(\frac{x_1 - x}{xz} \right) q(x_1/x) a(x_1) b(x) \\ & \quad - (xz)^{-1} \delta \left(\frac{x - x_1}{-xz} \right) q(x_1/x) K(x_1, x). \end{aligned}$$

Applying Res_z to both sides, we get

$$\begin{aligned} & q(x_1/x) (a(x_1) b(x) - K(x_1, x)) \\ &= \text{Res}_{x_0} q(x_1/x) Y_{\mathcal{E}}^e(a(x), x_0) b(x) e^{x_0 x \frac{\partial}{\partial x}} \delta \left(\frac{x}{x_1} \right) \\ &= \sum_{i \geq 0} a(x)_i^e b(x) \frac{1}{i!} q(x_1/x) \left(x \frac{\partial}{\partial x} \right)^i \delta \left(\frac{x}{x_1} \right), \end{aligned} \tag{4.20}$$

which is a finite sum. Combining this with (4.19) we obtain

$$\sum_{i \geq 0} (A_i(x) - a(x)_i b(x)) \frac{1}{i!} q(x_1/x) \left(x \frac{\partial}{\partial x} \right)^i \delta \left(\frac{x}{x_1} \right) = 0, \tag{4.21}$$

where we set $A_i(x) = 0$ for $i > r$. Write

$$q(x_1/x) = (x_1/x - 1)^{r+1} P(x_1/x) + \alpha,$$

where $P(x) \in \mathbb{C}[x]$ and

$$\alpha = (-1)^{k(s+1)} \prod_{i=1}^k (\lambda_i - 1)^{(r+1)(s+1)}.$$

Note that $\alpha \neq 0$ as $\lambda_i \neq 1$ for $1 \leq i \leq k$. Since

$$(x_1/x - 1)^{r+1} \left(x \frac{\partial}{\partial x} \right)^j \delta \left(\frac{x}{x_1} \right) = 0 \quad \text{for } 0 \leq j \leq r,$$

(4.21) reduces to

$$\sum_{i \geq 0} (A_i(x) - a(x)_i b(x)) \frac{1}{i!} \alpha \left(x \frac{\partial}{\partial x} \right)^i \delta \left(\frac{x}{x_1} \right) = 0, \quad (4.22)$$

which implies

$$\alpha (A_i(x) - a(x)_i b(x)) = 0 \quad \text{for } i \geq 0.$$

Then our assertions follow immediately. \square

Using Lemma 4.13 we have the following generalization:

Proposition 4.14. *Let W be a vector space, let $a(x), b(x) \in \mathcal{E}(W)$, and let*

$$K(x_1, x_2) \in \text{Hom}(W, W((x_2))((x_1))).$$

Assume

$$a(x_1)b(x_2) - K(x_1, x_2) = \sum_{i=1}^k \sum_{j=0}^r A_{ij}(x_2) \frac{1}{j!} \left(x_2 \frac{\partial}{\partial x_2} \right)^j \delta \left(\frac{\lambda_i x_2}{x_1} \right), \quad (4.23)$$

where $A_{ij}(x) \in \mathcal{E}(W)$ and $\lambda_i \in \mathbb{C}^\times$ distinct for $1 \leq i \leq k$. Then

$$\begin{aligned} a(\lambda_i x)_j^e b(x) &= A_{ij}(x) \quad \text{for } 0 \leq j \leq r, \\ a(\lambda_i x)_j^e b(x) &= 0 \quad \text{for } j > r. \end{aligned} \quad (4.24)$$

Proof. For $\lambda \in \mathbb{C}^\times$, from (4.23) we have

$$a(\lambda x_1)b(x_2) - K(\lambda x_1, x_2) = \sum_{i=1}^k \sum_{j=0}^r A_{ij}(x_2) \frac{1}{j!} \left(x_2 \frac{\partial}{\partial x_2} \right)^j \delta \left(\frac{\lambda_i x_2}{\lambda x_1} \right).$$

Taking $\lambda = \lambda_i$ with $1 \leq i \leq k$, using Lemma 4.13 we get

$$a(\lambda_i x)_j^e b(x) = A_{ij}(x) \quad \text{for } 0 \leq j \leq r$$

and $a(\lambda_i x)_j^e b(x) = 0$ for $j > r$, as desired. \square

Combining Lemma 2.3 with Proposition 4.14 we immediately have:

Corollary 4.15. *Let W be a vector space, let $a(x), b(x) \in \mathcal{E}(W)$, and let*

$$K(x_1, x_2) \in \text{Hom}(W, W((x_2))((x_1))).$$

Assume

$$p(x_1/x_2)a(x_1)b(x_2) = p(x_1/x_2)K(x_1, x_2), \quad (4.25)$$

where $p(x) = (x - \lambda_1)^{k_1} \cdots (x - \lambda_r)^{k_r}$ with $\lambda_1, \dots, \lambda_r$ distinct nonzero complex numbers and with $k_i \geq 1$. Then

$$a(x_1)b(x_2) - K(x_1, x_2) = \text{Res}_{x_0} \sum_{i=1}^r Y_{\mathcal{E}}^e(a(\lambda_i x), x_0) b(x) e^{x_0 \left(x \frac{\partial}{\partial x} \right)} \delta \left(\frac{\lambda_i x}{x_1} \right). \quad (4.26)$$

In the following we shall establish certain technical results.

Lemma 4.16. *Let $a(x), b(x) \in \mathcal{E}(W)$ and let $\lambda \in \mathbb{C}^\times$. Suppose that k is an integer and $p(x)$ is a polynomial with $p(\lambda) \neq 0$ such that*

$$(x_1/x_2 - \lambda)^k p(x_1/x_2) a(x_1) b(x_2) \in \text{Hom}(W, W((x_1, x_2))).$$

Then

$$a(\lambda x)_n^e b(x) = 0 \quad \text{for all } n \geq k. \quad (4.27)$$

Proof. From assumption we have

$$(x_1/x_2 - 1)^k p(\lambda x_1/x_2) a(\lambda x_1) b(x_2) \in \text{Hom}(W, W((x_1, x_2))).$$

Then

$$\begin{aligned} & z^k Y_{\mathcal{E}}^e(a(\lambda x), z) b(x) \\ &= z^k (e^z - 1)^{-k} p(\lambda e^z)^{-1} ((x_1/x - 1)^k p(\lambda x_1/x) a(\lambda x_1) b(x))|_{x_1 = x e^z}, \end{aligned}$$

where $p(\lambda e^z)^{-1}$ denotes the inverse of $p(\lambda e^z)$ in $\mathbb{C}((z))$. We have $z^k (e^z - 1)^{-k} \in \mathbb{C}[[z]]$ and $p(\lambda e^z)^{-1} \in \mathbb{C}[[z]]$ as $p(\lambda) \neq 0$. We know that

$$((x_1/x - 1)^k p(\lambda x_1/x) a(\lambda x_1) b(x))|_{x_1 = x e^z}$$

contains only nonnegative powers of z . Then it follows that $z^k Y_{\mathcal{E}}^e(a(\lambda x), z) b(x)$ involves only nonnegative powers of z . Thus, $a(\lambda x)_n^e b(x) = 0$ for all $n \geq k$. \square

Lemma 4.17. *Let $a(x), b(x) \in \mathcal{E}(W)$. Suppose that $p(x)$ is a nonzero polynomial such that*

$$p(x_1/x_2) a(x_1) b(x_2) \in \text{Hom}(W, W((x_1, x_2))).$$

Write $p(x) = (x - 1)^s q(x)$ with $s \in \mathbb{N}$, $q(x) \in \mathbb{C}[x]$ such that $q(1) \neq 0$. Then for $k \in \mathbb{Z}$,

$$(x_1/x_2 - 1)^k q(x_1/x_2) a(x_1) b(x_2) \in \text{Hom}(W, W((x_1, x_2))) \quad (4.28)$$

if and only if $a(x)_j^e b(x) = 0$ for $j \geq k$.

Proof. The “only if” part follows from Lemma 4.16. Now, assume that $a(x)_j^e b(x) = 0$ for $j \geq k$. We have

$$a(x_1) b(x) = \iota_{x_1, x}(1/p(x_1/x)) A(x_1, x)$$

for some $A(x_1, x) \in \text{Hom}(W, W((x_1, x)))$. If $A(x_1, x) = 0$, there is nothing to prove. Assume $A(x_1, x) \neq 0$. By Lemma 2.4 we have $A(x_1, x) = (x_1/x - 1)^r B(x_1, x)$ for some $r \in \mathbb{N}$, $B(x_1, x) \in \text{Hom}(W, W((x_1, x)))$ with $B(x, x) \neq 0$. Then

$$(x_1/x - 1)^{s-r} q(x_1/x) a(x_1) b(x) = B(x_1, x) \in \text{Hom}(W, W((x_1, x))), \quad (4.29)$$

so that

$$Y_{\mathcal{E}}^e(a(x), z)b(x) = (e^z - 1)^{r-s} q(e^z)^{-1} B(xe^z, x).$$

We have

$$\lim_{z \rightarrow 0} z^{s-r} Y_{\mathcal{E}}^e(a(x), z)b(x) = \lim_{z \rightarrow 0} \left(\frac{e^z - 1}{z} \right)^{r-s} q(e^z)^{-1} B(xe^z, x) = q(1)B(x, x).$$

We get $a(x)_n^e b(x) = 0$ for $n \geq s - r$ and $a(x)_{s-r-1}^e b(x) \neq 0$. As we are given that $a(x)_m^e b(x)$ for $m \geq k$, we must have $k \geq s - r$. Then it follows from (4.29) that (4.28) holds. \square

Definition 4.18. Let V be a T-type weak quantum vertex algebra and let G be an automorphism group equipped with a linear character $\chi : G \rightarrow \mathbb{C}^\times$. A G -covariant ϕ -coordinated quasi V -module is a ϕ -coordinated quasi V -module (W, Y_W) , satisfying the conditions that

$$Y_W(gv, x) = Y_W(v, \chi(g)x) \quad \text{for } g \in G, v \in V, \quad (4.30)$$

and that for $u, v \in V$, there exists $p(x) \in \mathbb{C}[x]$ with only zeroes in $\chi(G)$ such that

$$p(x_1/x_2)Y_W(u, x_1)Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))). \quad (4.31)$$

As the main result of this section we have:

Theorem 4.19. Let V, G, χ be given as in Definition 4.18 and let (W, Y_W) be a G -covariant ϕ -coordinated quasi V -module. Suppose that $u, v \in V$,

$$u^{(i)}, v^{(i)} \in V, f_i(x) \in \mathbb{C}(x) \quad (i = 1, \dots, r)$$

such that

$$(x_1 - x_2)^k Y(u, x_1)Y(v, x_2) = (x_1 - x_2)^k \sum_{i=1}^r \iota_{x_2, x_1} f_i(e^{x_1 - x_2}) Y(v^{(i)}, x_2) Y(u^{(i)}, x_1)$$

for some nonnegative integer k . Then there are finitely many $g_1, \dots, g_n \in G$ with $\chi(g_1), \dots, \chi(g_n)$ distinct such that

$$\begin{aligned} & Y_W(u, x_1)Y_W(v, x) - \sum_{i=1}^r \iota_{x_1, x} (f_i(x_1/x)) Y_W(v^{(i)}, x) Y_W(u^{(i)}, x_1) \\ &= \text{Res}_{x_0} \sum_{j=1}^n Y_W(Y(g_j(u), x_0)v, x) e^{x_0(x \frac{\partial}{\partial x})} \delta \left(\frac{\chi(g_j)x}{x_1} \right). \end{aligned} \quad (4.32)$$

Furthermore, if the linear character $\chi : G \rightarrow \mathbb{C}^\times$ is injective, we have

$$\begin{aligned} & Y_W(u, x_1)Y_W(v, x) - \sum_{i=1}^r \iota_{x_1, x} (f_i(x_1/x)) Y_W(v^{(i)}, x) Y_W(u^{(i)}, x_1) \\ &= \text{Res}_{x_0} \sum_{g \in G} Y_W(Y(gu, x_0)v, x) e^{x_0(x \frac{\partial}{\partial x})} \delta \left(\frac{\chi(g)x}{x_1} \right), \end{aligned} \quad (4.33)$$

which is a finite sum.

Proof. From definition, there are distinct nonzero complex numbers $\lambda_1, \dots, \lambda_n \in \chi(G)$ such that

$$p(x_1/x_2)Y_W(u, x_1)Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))),$$

where $p(x) = (x - \lambda_1)^{k_1} \cdots (x - \lambda_n)^{k_n}$ with $k_i \geq 1$. By Proposition 4.3 we have

$$\begin{aligned} & p(x_1/x_2)Y_W(u, x_1)Y_W(v, x_2) \\ &= p(x_1/x_2) \sum_{i=1}^r \iota_{x_1, x_2}(f_i(x_1/x_2)) Y_W(v^{(i)}, x_2) Y_W(u^{(i)}, x_1). \end{aligned}$$

Then by Corollary 4.15 we have

$$\begin{aligned} & Y_W(u, x_1)Y_W(v, x) - \sum_{i=1}^r \iota_{x_1, x}(f_i(x_1/x)) Y_W(v^{(i)}, x) Y_W(u^{(i)}, x_1) \\ &= \text{Res}_{x_0} \sum_{j=1}^n Y_{\mathcal{E}}^e \left(\overline{Y_W(u, \lambda_j x)}, x_0 \right) \overline{Y_W(v, x)} e^{x_0(x \frac{\partial}{\partial x})} \delta \left(\frac{\lambda_j x}{x_1} \right). \end{aligned} \quad (4.34)$$

For $1 \leq j \leq n$, let $g_j \in G$ be such that $\lambda_j = \chi(g_j)$. Then

$$Y_W(u, \lambda_j x) = Y_W(u, \chi(g_j)x) = Y_W(g_j u, x).$$

Using the fact that $Y_W(\cdot, x)$ is a homomorphism (from Proposition 4.11), we have

$$Y_{\mathcal{E}}^e \left(\overline{Y_W(u, \lambda_j x)}, x_0 \right) \overline{Y_W(v, x)} = Y_{\mathcal{E}}^e \left(\overline{Y_W(g_j u, x)}, x_0 \right) \overline{Y_W(v, x)} = Y_W(Y(g_j u, x_0)v, x).$$

Then we obtain

$$\begin{aligned} & Y_W(u, x_1)Y_W(v, x) - \sum_{i=1}^r \iota_{x_1, x}(f_i(x_1/x)) Y_W(v^{(i)}, x) Y_W(u^{(i)}, x_1) \\ &= \text{Res}_{x_0} \sum_{j=1}^n Y_W(Y(g_j u, x_0)v, x) e^{x_0(x \frac{\partial}{\partial x})} \delta \left(\frac{\lambda_j x}{x_1} \right), \end{aligned}$$

proving the first assertion.

Assume that χ is injective. For $g \in G$, if $g \neq g_j$ for $1 \leq j \leq n$, with $\chi(g) \neq \lambda_j$ we have $p(\chi(g)) \neq 0$. By Lemma 4.16 we get $Y_W(u, \chi(g)x)_m^e Y_W(v, x) = 0$ for $m \geq 0$. Then

$$Y_W(Y(gu, x_0)v, x) = Y_{\mathcal{E}}^e(Y_W(gu, x), x_0)Y_W(v, x) = Y_{\mathcal{E}}^e(Y_W(u, \chi(g)x), x_0)Y_W(v, x),$$

involving only nonnegative powers of x_0 . Thus

$$\text{Res}_{x_0} Y_W(Y(gu, x_0)v, x) e^{x_0(x \frac{\partial}{\partial x})} \delta \left(\frac{\chi(g)x}{x_1} \right) = 0.$$

Then the second assertion follows. \square

As an immediate consequence of Theorem 4.19 we have:

Corollary 4.20. *Let V be a T -type weak quantum vertex algebra and let (W, Y_W) be a G -covariant ϕ -coordinated quasi V -module. Let $u, v \in V$. Then*

$$p(x_1/x_2)Y_W(u, x_1)Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))), \quad (4.35)$$

where $p(x)$ is a product of $(x - \chi(\sigma))^{k_\sigma}$ with $\sigma \in G$ and $k_\sigma \geq 1$ such that

$$(\sigma u)_{k_\sigma-1}v \neq 0 \quad \text{and} \quad (\sigma u)_nv = 0 \quad \text{for } n \geq k_\sigma.$$

The following is a useful technical result:

Lemma 4.21. *Let V be a T -type weak quantum vertex algebra, let G be an automorphism group with a linear character χ , and let (W, Y_W) be a ϕ -coordinated quasi V -module. Suppose that U is a generating subset of V such that*

$$Y_W(gu, x) = Y_W(u, \chi(g)x) \quad \text{for } u \in U, g \in G \quad (4.36)$$

and such that $\{Y_W(u, x) \mid u \in U\}$ is $\chi(G)$ -quasi \mathcal{S}_{trig} -local. Then (W, Y_W) is a G -covariant ϕ -coordinated quasi V -module.

Proof. Set $K = \{v \in V \mid Y_W(gv, x) = Y_W(v, \chi(g)x) \text{ for } g \in G\}$. From assumption, we have $U \cup \{1\} \subset K$. Let $u, v \in K, g \in G$. As $Y_W(\cdot, x)$ is a homomorphism of weak quantum vertex algebras by Proposition 4.11, we get

$$\begin{aligned} Y_W(gY(u, z)v, x) &= Y_W(Y(gu, z)gv, x) \\ &= Y_{\mathcal{E}}^e(Y_W(gu, x), z)Y_W(gv, x) \\ &= Y_{\mathcal{E}}^e(Y_W(u, \chi(g)x), z)Y_W(v, \chi(g)x) \\ &= Y_W(Y(u, z)v, \chi(g)x) \end{aligned}$$

for $g \in G$. This shows $u_nv \in K$ for $n \in \mathbb{Z}$. Since U generates V , we have $V = K$, proving that (4.36) holds with V in place of U , i.e., (4.30) holds.

Recall Lemma 5.2 from [Li6] that every quasi \mathcal{S}_{trig} -local subset of $\mathcal{E}(W)$ is quasi compatible. Let Γ be a subgroup of \mathbb{C}^\times . We say that a finite sequence $\psi_1(x), \dots, \psi_r(x)$ in $\mathcal{E}(W)$ is Γ -quasi compatible if there exists a nonzero polynomial $p(x)$ with only zeros in Γ such that

$$\left(\prod_{1 \leq i < j \leq r} p(x_i/x_j) \right) \psi_1(x_1) \cdots \psi_r(x_r) \in \text{Hom}(W, W((x_1, \dots, x_r))). \quad (4.37)$$

Furthermore, a subset U of $\mathcal{E}(W)$ is Γ -quasi compatible if any finite sequence in U is Γ -quasi compatible. The proof of the very Lemma 5.2 with the obvious changes shows that every Γ -quasi \mathcal{S}_{trig} -local subset is Γ -quasi compatible.

On the other hand, recall Proposition 4.9 in [Li6]: Let

$$\psi_1(x), \dots, \psi_r(x), a(x), b(x), \phi_1(x), \dots, \phi_s(x) \in \mathcal{E}(W).$$

Assume that the ordered sequences $(a(x), b(x))$ and

$$(\psi_1(x), \dots, \psi_r(x), a(x), b(x), \phi_1(x), \dots, \phi_s(x))$$

are quasi compatible. Then for any $n \in \mathbb{Z}$, the ordered sequence

$$(\psi_1(x), \dots, \psi_r(x), a(x)_n^e b(x), \phi_1(x), \dots, \phi_s(x))$$

is quasi compatible. The same proof with the obvious changes shows that the refinement of Proposition 4.9 with quasi-compatibility replaced by Γ -quasi compatibility holds. It then follows from induction that for any Γ -quasi compatible subset A of $\mathcal{E}(W)$, $\langle A \rangle_e$ is Γ -quasi compatible.

Note that $Y_W(\cdot, x)$ is a homomorphism of weak quantum vertex algebras from V to V_W where $V_W = \{Y_W(v, x) \mid v \in V\}$. Since U is a generating subset of V , it follows that $V_W = \langle U_W \rangle_e$ with $U_W = \{Y_W(u, x) \mid u \in U\}$. Then V_W is $\chi(G)$ -quasi compatible. In particular, (4.31) holds. Therefore, (W, Y_W) is a G -covariant ϕ -coordinated quasi V -module. \square

As we need, recall the following result from [Li6] (Lemma 6.7):

Lemma 4.22. *Let W be a vector space and let $a(x), b(x) \in \mathcal{E}(W)$. Assume that there exist*

$$0 \neq p(x) \in \mathbb{C}[x], \quad q_i(x) \in \mathbb{C}((x)), \quad u^{(i)}(x), v^{(i)}(x) \in \mathcal{E}(W) \quad (1 \leq i \leq r)$$

such that

$$p(x_1/x_2)a(x_1)b(x_2) = \sum_{i=1}^r q_i(x_1/x_2)u^{(i)}(x_2)v^{(i)}(x_1). \quad (4.38)$$

Then $(a(x), b(x))$ is quasi compatible and

$$\begin{aligned} & p(e^{x_0})Y_{\mathcal{E}}^e(a(x), x_0)b(x) \\ = & \text{Res}_{x_1} \left(\frac{1}{x_1 - xe^{x_0}} p(x_1/x)a(x_1)b(x) - \frac{1}{-xe^{x_0} + x_1} \sum_{i=1}^r q_i(x_1/x)u^{(i)}(x)v^{(i)}(x_1) \right). \end{aligned}$$

Furthermore, if k is the order of zero of $p(x)$ at 1, then $a(x)_n^e b(x) = 0$ for $n \geq k$ and

$$\begin{aligned} & \frac{1}{k!} p^{(k)}(1)a(x)_{k-1}^e b(x) \\ = & \text{Res}_{x_1} \left(\frac{1}{x_1 - x} p(x_1/x)a(x_1)b(x) - \frac{1}{-x + x_1} q(x_1/x)u^{(i)}(x)v^{(i)}(x_1) \right). \quad (4.39) \end{aligned}$$

5 Deformed Virasoro algebra $\mathcal{V}ir_{p,q}$ with $q = -1$

In this section, we study the deformed Virasoro algebra $\mathcal{V}ir_{p,q}$ with $q = -1$ in the context of quantum vertex algebras and ϕ -coordinated quasi modules. We present a canonical connection of this algebra with a Clifford vertex super-algebra and its G -covariant ϕ -coordinated quasi modules.

We first recall from [SKAO] the definition of the deformed Virasoro algebra $\mathcal{V}ir_{p,q}$. Let p and q be nonzero complex numbers with p not a root of -1 , so that $p^n + 1 \neq 0$ for every integer n . Set

$$t = \frac{q}{p}.$$

Definition 5.1. The deformed Virasoro algebra $\mathcal{V}ir_{p,q}$ is defined to be the associative unital algebra over \mathbb{C} with generators T_n ($n \in \mathbb{Z}$), subject to relations

$$\sum_{l=0}^{\infty} f_l (T_{m-l} T_{n+l} - T_{n-l} T_{m+l}) = -\frac{(1-q)(1-p/q)}{1-p} (p^m - p^{-m}) \delta_{m+n,0} \quad (5.1)$$

for $m, n \in \mathbb{Z}$, where the coefficients f_l 's are given by

$$f(z) \equiv \sum_{l=0}^{\infty} f_l z^l = \exp \left(\sum_{n=1}^{\infty} \frac{(1-q^n)(1-t^{-n})}{1+p^n} \frac{z^n}{n} \right). \quad (5.2)$$

Remark 5.2. Note that in general, $f_l \neq 0$ for infinitely many l , so that the expression on the left-hand side of (5.1) is a genuine infinite sum. Because of this, the algebra $\mathcal{V}ir_{p,q}$ is a topological algebra (involving a formal completion). This can be done rigorously by imitating Frenkel-Zhu's construction of the universal enveloping algebra $U(V)$ of a vertex operator algebra V (see [FZ]).

Form a generating function

$$T(x) = \sum_{n \in \mathbb{Z}} T_n x^{-n}. \quad (5.3)$$

Then the defining relations (5.1) can be written as

$$\begin{aligned} & f(z/x) T(x) T(z) - f(x/z) T(z) T(x) \\ &= -\frac{(1-q)(1-p/q)}{1-p} \left[\delta \left(\frac{pz}{x} \right) - \delta \left(\frac{z}{px} \right) \right]. \end{aligned} \quad (5.4)$$

From the defining relations (5.1), it can be readily seen that $\mathcal{V}ir_{p,q}$ is a \mathbb{Z} -graded algebra with $\deg T_n = -n$ for $n \in \mathbb{Z}$. Then one can study highest weight $\mathcal{V}ir_{p,q}$ -modules, in a way similar to that for the ordinary Virasoro algebra, and in fact, the highest weight $\mathcal{V}ir_{p,q}$ -modules have been studied in [SKAO] and [BP]. One can also define restricted modules, which are more general than highest weight modules.

The deformed Virasoro algebra $\mathcal{V}ir_{p,q}$ with $q = -1$ has been extensively studied in [BP] from an associative algebra point of view. In this case, one has

$$f(z) = \frac{1+z}{1-z} = 1 + 2 \sum_{l=1}^{\infty} z^l.$$

The defining relations for $\mathcal{V}ir_{p,-1}$ read as

$$[T_m, T_n] + 2 \sum_{l \geq 0} (T_{m-l} T_{n+l} - T_{n-l} T_{m+l}) = -2 \left(\frac{1+p}{1-p} \right) (p^m - p^{-m}) \delta_{m+n,0} \quad (5.5)$$

for $m, n \in \mathbb{Z}$. In terms of the generating function $T(x) = \sum_{n \in \mathbb{Z}} T_n x^{-n}$, we have

$$\begin{aligned} & \left(\frac{1 + \frac{x_2}{x_1}}{1 - \frac{x_2}{x_1}} \right) T(x_1) T(x_2) - \left(\frac{1 + \frac{x_1}{x_2}}{1 - \frac{x_1}{x_2}} \right) T(x_2) T(x_1) \\ &= -2 \left(\frac{1+p}{1-p} \right) \left(\delta \left(\frac{px_2}{x_1} \right) - \delta \left(\frac{p^{-1}x_2}{x_1} \right) \right). \end{aligned} \quad (5.6)$$

We say a $\mathcal{V}ir_{p,-1}$ -module W is *restricted* if for any $w \in W$, $T_n w = 0$ for n sufficiently large, namely, $T(x) \in \mathcal{E}(W)$.

Throughout this section, we assume that p is not a root of unity. Set

$$\Gamma_p = \{p^n \mid n \in \mathbb{Z}\} \subset \mathbb{C}^\times. \quad (5.7)$$

Next, we introduce a vertex super-algebra.

Definition 5.3. Let E be a complex vector space with a basis $\{e^{(r)} \mid r \in \mathbb{Z}\}$ and equip E with a bilinear form $\langle \cdot, \cdot \rangle$ defined by

$$\langle e^{(r)}, e^{(s)} \rangle = 2(\delta_{r,s+1} + \delta_{r,s-1}) \quad \text{for } r, s \in \mathbb{Z}. \quad (5.8)$$

It can be readily seen that this form is symmetric and non-degenerate. Set

$$L(E) = E \otimes \mathbb{C}[t, t^{-1}].$$

Let ℓ be a complex number. Define a bilinear form $\langle \cdot, \cdot \rangle_\ell$ on $L(E)$ by

$$\langle e^{(r)} \otimes t^m, e^{(s)} \otimes t^n \rangle_\ell = 2\ell(\delta_{r,s+1} + \delta_{r,s-1}) \delta_{m+n+1,0} \quad (5.9)$$

for $r, s, m, n \in \mathbb{Z}$. This form is symmetric and non-degenerate whenever $\ell \neq 0$. Denote by $\text{Cl}(L(E), \ell)$ the associated Clifford algebra, which by definition is the associative unital algebra generated by vector space $L(E)$, subject to relations

$$(a \otimes t^m)(b \otimes t^n) + (b \otimes t^n)(a \otimes t^m) = \langle a, b \rangle \delta_{m+n+1,0} \ell \quad (5.10)$$

for $a, b \in E$, $m, n \in \mathbb{Z}$.

Remark 5.4. The Clifford algebra $\text{Cl}(L(E), \ell)$ can be defined alternatively as the quotient algebra of the universal enveloping algebra of the Lie super-algebra

$$\hat{E} = (E \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}\mathbf{k},$$

modulo relation $\mathbf{k} = \ell$, where $\hat{E}^0 = \mathbb{C}\mathbf{k}$, $\hat{E}^1 = E \otimes \mathbb{C}[t, t^{-1}]$, and $[\mathbf{k}, \hat{E}] = 0$,

$$[a \otimes t^m, b \otimes t^n] = \langle a, b \rangle \delta_{m+n+1, 0} \mathbf{k}$$

for $a, b \in E$, $m, n \in \mathbb{Z}$.

Following the tradition, for $a \in E$, $n \in \mathbb{Z}$ we alternatively denote $a \otimes t^n$ by a_n . For $a \in E$, set

$$a(x) = \sum_{n \in \mathbb{Z}} a_n x^{-n-1} \in \text{Cl}(L(E), \ell)[[x, x^{-1}]]. \quad (5.11)$$

Set

$$V_{\hat{E}}(\ell, 0) = \text{Cl}(L(E), \ell) / \text{Cl}(L(E), \ell)(E \otimes \mathbb{C}[t]), \quad (5.12)$$

a left $\text{Cl}(L(E), \ell)$ -module, and set

$$\mathbf{1} = 1 + \text{Cl}(L(E), \ell)(E \otimes \mathbb{C}[t]) \in V_{\hat{E}}(\ell, 0).$$

Identify E as a subspace of $V_{\hat{E}}(\ell, 0)$ through the map $a \mapsto a(-1)\mathbf{1}$. It is well known (see [FFR]) that there exists a vertex super-algebra structure on $V_{\hat{E}}(\ell, 0)$, which is uniquely determined by the condition that $\mathbf{1}$ is the vacuum vector and $Y(a, x) = a(x)$ for $a \in E$.

As we need later, we mention some simple facts about $V_{\hat{E}}(\ell, 0)$. The subspace E generates $V_{\hat{E}}(\ell, 0)$ as a vertex super-algebra. Furthermore, for $r, s \in \mathbb{Z}$, we have

$$e_n^{(r)} e^{(s)} = \delta_{n,0} \ell \langle e^{(r)}, e^{(s)} \rangle \mathbf{1} = 2\ell \delta_{n,0} (\delta_{r,s+1} + \delta_{r,s-1}) \mathbf{1} \quad \text{for } n \geq 0. \quad (5.13)$$

We also have

$$e_{-1}^{(r)} e^{(r)} = 0 \quad \text{for } r \in \mathbb{Z}. \quad (5.14)$$

Lemma 5.5. *Assume that ℓ is a nonzero complex number. Then $V_{\hat{E}}(\ell, 0)$ viewed as a $\text{Cl}(L(E), \ell)$ -module is irreducible and $V_{\hat{E}}(\ell, 0)$ viewed as a vertex super-algebra is simple.*

Proof. Let k be any positive integer. Set $E[k] = \text{span}\{e^{(r)} \mid -k \leq r \leq k+1\}$ (even dimensional). One can show that the bilinear form of E , restricted to $E[k]$, is non-degenerate. Let $V[k]$ denote the vertex super subalgebra of $V_{\hat{E}}(\ell, 0)$, generated by $E[k]$. From [FFR], $V[k]$ is an irreducible $\text{Cl}(L(E[k]), \ell)$ -module. It is clear that $V[k]$ with $k \geq 1$ form an increasing filtration of $V_{\hat{E}}(\ell, 0)$. Then it follows that $V_{\hat{E}}(\ell, 0)$ is an irreducible $\text{Cl}(L(E), \ell)$ -module. This particularly implies that $V_{\hat{E}}(\ell, 0)$ is a simple vertex super-algebra. \square

For the rest of this section we consider $\ell = 1$. The following is straightforward:

Lemma 5.6. *For $m \in \mathbb{Z}$, there exists an automorphism σ_m of $V_{\hat{E}}(1, 0)$, which is uniquely determined by*

$$\sigma_m(e^{(r)}) = e^{(m+r)} \quad \text{for } r \in \mathbb{Z}.$$

Furthermore, this gives rise to a group action of \mathbb{Z} on $V_{\hat{E}}(1, 0)$ by automorphisms.

Define a linear character of \mathbb{Z} :

$$\chi_p : \mathbb{Z} \rightarrow \mathbb{C}^\times \quad \text{with } \chi_p(n) = p^n \text{ for } n \in \mathbb{Z}. \quad (5.15)$$

Now, we are in a position to present our first main result of this section.

Theorem 5.7. *Let W be any restricted $\text{Vir}_{p,-1}$ -module. Then there exists a (\mathbb{Z}, χ_p) -covariant ϕ -coordinated quasi $V_{\hat{E}}(1, 0)$ -module structure $Y_W(\cdot, x)$ which is uniquely determined by the condition*

$$Y_W(e^{(r)}, x) = T(p^r x) \quad \text{for } r \in \mathbb{Z}.$$

Proof. As E generates $V_{\hat{E}}(1, 0)$ as a vertex super-algebra, the uniqueness is clear. We now prove the existence. Set

$$U_W = \{T(p^m x) \mid m \in \mathbb{Z}\} \subset \mathcal{E}(W).$$

From (5.6) we have

$$\begin{aligned} & \left(\frac{x_1 + p^{s-r} x_2}{x_1 - p^{s-r} x_2} \right) T(p^r x_1) T(p^s x_2) - \left(\frac{x_2 + p^{r-s} x_1}{x_2 - p^{r-s} x_1} \right) T(p^s x_2) T(p^r x_1) \\ &= -2 \left(\frac{1+p}{1-p} \right) \left(\delta \left(\frac{p^{s+1-r} x_2}{x_1} \right) - \delta \left(\frac{p^{s-1-r} x_2}{x_1} \right) \right) \end{aligned} \quad (5.16)$$

for $r, s \in \mathbb{Z}$. By multiplying both sides of (5.16) by a suitable polynomial we get

$$\begin{aligned} & (x_1/x_2 - p^{s+1-r})(x_1/x_2 - p^{s-1-r})(x_1/x_2 + p^{s-r}) T(p^r x_1) T(p^s x_2) \\ &= -(x_1/x_2 - p^{s+1-r})(x_1/x_2 - p^{s-1-r})(x_1/x_2 + p^{s-r}) T(p^s x_2) T(p^r x_1). \end{aligned} \quad (5.17)$$

From definition, U_W is Γ_p -stable and from (5.17), U_W is quasi \mathcal{S}_{trig} -local. In view of Theorem 4.10, U_W generates a weak quantum vertex algebra $\langle U_W \rangle_e$ with W as a faithful ϕ -coordinated quasi module.

We next show that $\langle U_W \rangle_e$ becomes a $\text{Cl}(L(E), 1)$ -module by letting $e^{(r)}(z)$ act as $Y_{\mathcal{E}}^e(T(p^r x), z)$ for $r \in \mathbb{Z}$. To this end, we need to prove

$$\begin{aligned} & Y_{\mathcal{E}}^e(T(p^r x), x_1) Y_{\mathcal{E}}^e(T(p^s x), x_2) + Y_{\mathcal{E}}^e(T(p^s x), x_2) Y_{\mathcal{E}}^e(T(p^r x), x_1) \\ &= 2(\delta_{r,s+1} + \delta_{s,r+1}) x_1^{-1} \delta \left(\frac{x_2}{x_1} \right) \end{aligned} \quad (5.18)$$

for $r, s \in \mathbb{Z}$. With (5.17), by Proposition 5.3 of [Li6], we have

$P(e^{x_1-x_2})Y_{\mathcal{E}}^e(T(p^r x), x_1)Y_{\mathcal{E}}^e(T(p^s x), x_2) = -P(e^{x_1-x_2})Y_{\mathcal{E}}^e(T(p^s x), x_2)Y_{\mathcal{E}}^e(T(p^r x), x_1)$,
where $P(x) = (x-p^{s+1-r})(x-p^{s-1-r})(x+p^{s-r})$. As p is not a root of -1 , $e^{x_1-x_2}+p^{s-r}$ is invertible in $\mathbb{C}[[x_1, x_2]]$. By cancellation we get

$$\begin{aligned} & (e^{x_1-x_2} - p^{s+1-r})(e^{x_1-x_2} - p^{s-1-r})Y_{\mathcal{E}}^e(T(p^r x), x_1)Y_{\mathcal{E}}^e(T(p^s x), x_2) \\ &= -(e^{x_1-x_2} - p^{s+1-r})(e^{x_1-x_2} - p^{s-1-r})Y_{\mathcal{E}}^e(T(p^s x), x_2)Y_{\mathcal{E}}^e(T(p^r x), x_1). \end{aligned} \quad (5.19)$$

If $r-s \neq \pm 1$, $(e^{x_1-x_2} - p^{s+1-r})$ and $(e^{x_1-x_2} - p^{s-1-r})$ are also invertible in $\mathbb{C}[[x_1, x_2]]$, so that we furthermore obtain

$$Y_{\mathcal{E}}^e(T(p^r x), x_1)Y_{\mathcal{E}}^e(T(p^s x), x_2) = -Y_{\mathcal{E}}^e(T(p^s x), x_2)Y_{\mathcal{E}}^e(T(p^r x), x_1). \quad (5.20)$$

This proves that (5.18) holds for $r \neq s \pm 1$.

Consider the case $r = s \pm 1$. Set $h(x) = \sum_{n \geq 1} \frac{1}{n!} x^{n-1}$. Then $e^x - 1 = xh(x)$. If $r = s + 1$, we have

$$(e^{x_1-x_2} - p^{s+1-r})(e^{x_1-x_2} - p^{s-1-r}) = (x_1 - x_2)h(x_1 - x_2)(e^{x_1-x_2} - p^{-2}),$$

where $h(x_1 - x_2)(e^{x_1-x_2} - p^{-2})$ is invertible in $\mathbb{C}[[x_1, x_2]]$. From (5.19) we get

$$\begin{aligned} & (x_1 - x_2)Y_{\mathcal{E}}^e(T(p^r x), x_1)Y_{\mathcal{E}}^e(T(p^s x), x_2) \\ &= -(x_1 - x_2)Y_{\mathcal{E}}^e(T(p^s x), x_2)Y_{\mathcal{E}}^e(T(p^r x), x_1). \end{aligned} \quad (5.21)$$

If $r = s - 1$, similarly we obtain the same relation. As $\langle U_W \rangle_e$ as a weak quantum vertex algebra is generated by $T(p^m x)$ for $m \in \mathbb{Z}$, it follows that $\langle U \rangle_e$ is actually a vertex super-algebra.

Recall that $P(x) = (x - p^{s+1-r})(x - p^{s-1-r})(x + p^{s-r})$. For $r = s + 1$, we have $P(x) = (x - 1)(x - p^{-2})(x + p^{-1})$ and $P'(1) = (1 - p^{-2})(1 + p^{-1})$. With (5.17), by Lemma 4.22 we have

$$T(p^r x)_n^e T(p^s x) = 0 \quad \text{for } n \geq 1$$

and

$$\begin{aligned} & P'(1)T(p^r x)_0^e T(p^s x) \\ &= \text{Res}_{x_1} \left(\frac{1}{x_1 - x} P(x_1/x) T(p^r x_1) T(p^s x) + \frac{1}{-x + x_1} P(x_1/x) T(p^s x) T(p^r x_1) \right) \\ &= \text{Res}_{x_1} x^{-1} (x_1/x - p^{-2})(x_1/x - p^{-1}) \cdot \\ & \quad \cdot \left(\left(\frac{x_1 + p^{-1}x}{x_1 - p^{-1}x} \right) T(p^r x_1) T(p^s x) - \left(\frac{p^{-1}x + x_1}{p^{-1}x - x_1} \right) T(p^s x) T(p^r x_1) \right) \\ &= \text{Res}_{x_1} x^{-1} (x_1/x - p^{-2})(x_1/x - p^{-1}) \cdot \\ & \quad \cdot (-2) \left(\frac{1+p}{1-p} \right) \left(\delta \left(\frac{x}{x_1} \right) - \delta \left(\frac{p^{-2}x}{x_1} \right) \right) \\ &= (1 - p^{-2})(1 - p^{-1})(-2) \left(\frac{1+p}{1-p} \right) \\ &= 2(1 - p^{-2})(1 + p^{-1})(\delta_{r,s+1} + \delta_{s,r+1}), \end{aligned}$$

which implies

$$T(p^r x)_0^e T(p^s x) = 2 \cdot 1_W.$$

It follows from the super-commutator formula that (5.18) holds for $r = s + 1$. Similarly, one can show that (5.18) holds for $r = s - 1$. Thus $\langle U_W \rangle_e$ is a $\text{Cl}(L(E), 1)$ -module with $e^{(r)}(z)$ acting as $Y_{\mathcal{E}}^e(T(p^r x), z)$ for $r \in \mathbb{Z}$. Furthermore, we have

$$e_n^{(r)} 1_W = T(p^r x)_n^e 1_W = 0 \quad \text{for } r \in \mathbb{Z}, n \in \mathbb{N}.$$

It follows from the construction of $V_{\hat{E}}(1, 0)$ that there exists a $\text{Cl}(L(E), 1)$ -module homomorphism ψ from $V_{\hat{E}}(1, 0)$ to $\langle U_W \rangle_e$, sending $\mathbf{1}$ to 1_W . That is,

$$\psi(Y(e^{(r)}, z)v) = Y_{\mathcal{E}}^e(T(p^r x), z)\psi(v) \quad \text{for } r \in \mathbb{Z}, v \in V_{\hat{E}}(1, 0).$$

Since E generates $V_{\hat{E}}(1, 0)$ as a vertex super-algebra, it follows that ψ is a homomorphism of vertex super-algebras. As W is a ϕ -coordinated quasi module for $\langle U \rangle_e$, W becomes a ϕ -coordinated quasi $V_{\hat{E}}(1, 0)$ -module through homomorphism ψ . Furthermore, we have

$$Y_W(\sigma_n(e^{(r)}), x) = Y_W(e^{(r+n)}, x) = T(p^{r+n} x) = Y_W(e^{(r)}, p^n x) = Y_W(e^{(r)}, \chi_p(n)x)$$

for $n, r \in \mathbb{Z}$. As E generates $V_{\hat{E}}(1, 0)$, by Lemma 4.21, it is also (\mathbb{Z}, χ_p) -covariant. Therefore, W is a (\mathbb{Z}, χ_p) -covariant ϕ -coordinated quasi $V_{\hat{E}}(1, 0)$ -module. \square

On the other hand, we have:

Theorem 5.8. *Let (W, Y_W) be a (\mathbb{Z}, χ_p) -covariant ϕ -coordinated quasi $V_{\hat{E}}(1, 0)$ -module. Then W is a restricted module for $\text{Vir}_{p,-1}$ with $T(x) = Y_W(e^{(1)}, x)$.*

Proof. As $Y_W(e^{(1)}, x) \in \mathcal{E}(W)$ from definition, we must prove

$$\begin{aligned} & \left(\frac{x_1 + x_2}{x_1 - x_2} \right) Y_W(e^{(1)}, x_1) Y_W(e^{(1)}, x_2) - \left(\frac{x_2 + x_1}{x_2 - x_1} \right) Y_W(e^{(1)}, x_2) Y_W(e^{(1)}, x_1) \\ &= -2 \left(\frac{1+p}{1-p} \right) \left(\delta \left(\frac{px_2}{x_1} \right) - \delta \left(\frac{p^{-1}x_2}{x_1} \right) \right). \end{aligned} \quad (5.22)$$

Note that for $n \in \mathbb{Z}$, $i \geq 0$, we have

$$(\sigma_n e^{(1)})_i e^{(1)} = e_i^{(n+1)} e^{(1)} = \delta_{i,0} \langle e^{(n+1)}, e^{(1)} \rangle \mathbf{1} = 2\delta_{i,0} (\delta_{n,1} + \delta_{n,-1}) \mathbf{1}.$$

In view of Theorem 4.19, we have

$$\begin{aligned} & Y_W(e^{(1)}, x_1) Y_W(e^{(1)}, x_2) + Y_W(e^{(1)}, x_2) Y_W(e^{(1)}, x_1) \\ &= 2 \left(\delta \left(\frac{px_2}{x_1} \right) + \delta \left(\frac{p^{-1}x_2}{x_1} \right) \right), \end{aligned} \quad (5.23)$$

noticing that χ_p is one-to-one. From this we obtain

$$\begin{aligned} & (x_1 - px_2)(px_1 - x_2)Y_W(e^{(1)}, x_1)Y_W(e^{(1)}, x_2) \\ &= -(x_1 - px_2)(px_1 - x_2)Y_W(e^{(1)}, x_2)Y_W(e^{(1)}, x_1), \end{aligned}$$

which implies

$$(x_1 - px_2)(px_1 - x_2)Y_W(e^{(1)}, x_1)Y_W(e^{(1)}, x_2) \in \text{Hom}(W, W((x_1, x_2))). \quad (5.24)$$

Note that $e_j^{(1)}e^{(1)} = 0$ for $j \geq -1$. Using the fact that Y_W is a homomorphism (from Proposition 4.11), we get

$$Y_W(e^{(1)}, x)_j^e Y_W(e^{(1)}, x) = Y_W(e_j^{(1)}e^{(1)}, x) = 0 \quad \text{for } j \geq -1.$$

From Lemma 4.17 we have

$$(x_1 - x_2)^{-1}(x_1 - px_2)(px_1 - x_2)Y_W(e^{(1)}, x_1)Y_W(e^{(1)}, x_2) \in \text{Hom}(W, W((x_1, x_2))).$$

Thus

$$\begin{aligned} (x_1 - px_2)(px_1 - x_2)Y_W(e^{(1)}, x_1)Y_W(e^{(1)}, x_2) &= (x_1 - x_2)A(x_1, x_2), \\ (x_1 - px_2)(px_1 - x_2)Y_W(e^{(1)}, x_2)Y_W(e^{(1)}, x_1) &= (x_2 - x_1)A(x_1, x_2) \end{aligned} \quad (5.25)$$

for some $A(x_1, x_2) \in \text{Hom}(W, W((x_1, x_2)))$. Then

$$\begin{aligned} & Y_W(e^{(1)}, x_1)Y_W(e^{(1)}, x_2) + Y_W(e^{(1)}, x_2)Y_W(e^{(1)}, x_1) \\ &= \left(\frac{x_1 - x_2}{(x_1 - px_2)(px_1 - x_2)} + \frac{x_2 - x_1}{(x_2 - px_1)(px_2 - x_1)} \right) A(x_1, x_2) \\ &= \frac{1}{p+1} \left(x_1^{-1} \delta \left(\frac{px_2}{x_1} \right) + x_2^{-1} \delta \left(\frac{px_1}{x_2} \right) \right) A(x_1, x_2). \end{aligned} \quad (5.26)$$

Comparing this with (5.23) we get

$$\begin{aligned} x_1^{-1} \delta \left(\frac{px_2}{x_1} \right) A(x_1, x_2) &= 2(p+1) \delta \left(\frac{px_2}{x_1} \right), \\ x_2^{-1} \delta \left(\frac{px_1}{x_2} \right) A(x_1, x_2) &= 2(p+1) \delta \left(\frac{px_1}{x_2} \right). \end{aligned}$$

Using these identities we obtain

$$\begin{aligned} & \left(\frac{x_1 + x_2}{x_1 - x_2} \right) Y_W(e^{(1)}, x_1)Y_W(e^{(1)}, x_2) - \left(\frac{x_2 + x_1}{x_2 - x_1} \right) Y_W(e^{(1)}, x_2)Y_W(e^{(1)}, x_1) \\ &= \left(\frac{x_1 + x_2}{(x_1 - px_2)(px_1 - x_2)} - \frac{x_1 + x_2}{(px_2 - x_1)(x_2 - px_1)} \right) A(x_1, x_2) \\ &= \frac{1}{p-1} \left(\frac{1}{x_1 - px_2} - \frac{1}{px_1 - x_2} + \frac{1}{px_2 - x_1} - \frac{1}{x_2 - px_1} \right) A(x_1, x_2) \\ &= \frac{1}{p-1} \left(x_1^{-1} \delta \left(\frac{px_2}{x_1} \right) - x_2^{-1} \delta \left(\frac{px_1}{x_2} \right) \right) A(x_1, x_2) \\ &= -2 \left(\frac{1+p}{1-p} \right) \left(\delta \left(\frac{px_2}{x_1} \right) - \delta \left(\frac{p^{-1}x_2}{x_1} \right) \right). \end{aligned}$$

Therefore, W is a restricted module for $\mathcal{V}ir_{p,-1}$ with $T(x) = Y_W(e^{(1)}, x)$. \square

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